ANALYTIC GEOMETRY

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ANALYTIC GEOMETRY

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PREFACE

In the preparation of this book the author has tried to keep in mind the twofold requirement of a text-book on Analytic Geometry: to bring out clearly the fundamental principles and methods of the subject, and to make it a natural introduction to more advanced work. Since for most students of Analytic Geometry the subject is quite as essential as a preparation for the study of Calculus as it is valuable for its own methods and body of facts, the method and notation of the Calculus have been used in their application to tangents, normals, and maxima and minima in the plane, and to tangent planes and lines in space.

The conic sections have not been accorded as much space relatively as in most text-books on the subject, but it is believed that the student's time in the usual brief course can be spent to greater profit in the study of such chapters as those on Trigonometric and Exponential Functions, Parametric Equations, Empirical Equations, Maxima and Minima, and Graphical Solution of Equations, than upon a prolonged course on the conics. Especially is this true for engineering students.

The answers to many of the problems have not been given. Where the student can check the answer by graphical means, it is best that he should thus test the correctness of his work, and a complete list of answers tends to take away his incentive for doing this.

The author is under many obligations to Professors D. F. Campbell, Alexander Pell, C. W. Leigh, and C. I. Palmer, of the Armour Institute of Technology, and to Mr. Paul Dorweiler

of the Carnegie Technical Schools, for valuable criticism and advice, and to Professors Leigh and Palmer for the answers to many of the problems. The imperfections of the book are, however, the author's alone.

For the drawing of most of the figures the author is indebted to Mr. John R. Boyd, and for the remainder to Mr. Edwin O. Kaul, students in the School of Applied Science, Carnegie Technical Schools.

N. C. RIGGS.

CARNEGIE TECHNICAL SCHOOLS,
August, 1910.

CONTENTS

CHAPTER IV

Articles 46–50

Locus of an Equation

CHAPTER V

Articles 51–54

Transformation of Coördinates

CHAPTER VI

Articles 55–71

The Straight Line

CHAPTER VII

Articles 72–96

Standard Equations of Second Degree

CHAPTER VIII

Articles 97–107

TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

CHAPTER IX

Articles 108–120

PARAMETRIC EQUATIONS OF LOCI

CHAPTER X

Articles 121–124

INTERSECTIONS OF CURVES. SLOPE EQUATIONS OF TANGENTS

CHAPTER XI

Articles 125–154

SLOPES. TANGENTS AND NORMALS. DERIVATIVES

CHAPTER XII

Articles 155–163

MAXIMA AND MINIMA. DERIVATIVE CURVES

CHAPTER XIII

Articles 164–177

The Conic Sections

CHAPTER XIV

Articles 178–186

Properties of Conics

CHAPTER XV

Articles 187–195

The General Equation of Second Degree

CHAPTER XVI

Articles 196–201

Empirical Equations

CHAPTER XVII

Articles 202–212

Coördinates in Space

CHAPTER XVIII

Articles 213–217

Loci and their Equations

CHAPTER XIX

Articles 218–225

The Plane and the Straight Line

CHAPTER XX

Articles 226–234

The Quadric Surfaces

CHAPTER XXI

Articles 235–239

Space Curves

CHAPTER XXII

Articles 240–246

Tangent Lines and Planes

ANALYTIC GEOMETRY

CHAPTER I

GRAPHICAL REPRESENTATION OF NUMBERS. SYSTEMS OF COÖRDINATES

I. POINTS ON A STRAIGHT LINE

1. Point and Number. On a straight line let a fixed point O be taken from which to measure distances, and let a definite length be chosen as a unit. If this unit be laid off in succession on the line, beginning at O, other points of the line are obtained whose distances from O are 1, 2, 3, $\cdots$, etc. times the unit distance. It is convenient to think of these points as representing the numbers, or of the numbers as representing the points.

Thus a point 7 units from O may be taken to represent the number 7, and conversely the number 7 may be said to represent the point.

Since there are two points of the line at the same distance from P, one to the right, the other to the left, and since there are both positive and negative numbers, let it be agreed that points to the right of O shall represent positive numbers and those to the left of O negative numbers.

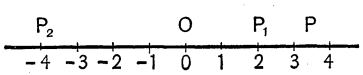


FIG. 1.

Thus a point 3 units to the right of O represents the number 3, and a point 3 units to the left of O represents the number -3. The numbers are also said to represent the points.

It can be shown that to every point of the line there corresponds a real number, and conversely, to every real number there corresponds a point of the line. The whole system of real numbers may therefore be represented by points on a

straight line with one number for each point and one point for each number.

The point O is called the **origin**. It represents the number zero.

2. Notation. If P is any point of the line and O is the origin, the symbol OP is used to denote the number which represents the point P.

E.g. if P lies 3 units to the right of O, then OP is 3; while if P lies 3 units to the left of O, OP is -3.

It is convenient to denote the number which represents a point by a single letter, as x; thus $OP = x$. Then if P lies to the right of O, x is a positive number, and if P lies to the left of O, x is a negative number.

Different points on the line will sometimes be denoted by P with different subscripts, and the numbers representing these points by x with corresponding subscripts.

Thus, in Fig. 1, $OP_1 = x_1 = 2$, $OP_2 = x_2 = -4$.

3. Segments of the line. In speaking of any segment of the line, as AB, the first letter named is called the **beginning**, and the last letter the **end**, of the segment.

Thus A is the beginning, and B is the end, of AB, while B is the beginning, and A is the end, of BA.

It is important to represent the value of any segment of the line by a number, and this is done by defining the value of any segment of the line to be the number which would represent the end of the segment if the beginning of the segment were taken as origin.

Thus, in Fig. 2, with O as origin,

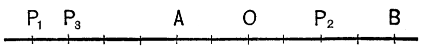


FIG. 2.

$AB = 6$, $BA = -6$, $P_3O = 5$, $OP_1 = -6$, $OA = -2$, $P_3P_1 = -1$, $P_2O = -2$.

From the definition of the value of a segment it follows that the value of any segment read from right to left is negative, while the value of any segment read from left to right is positive.

EXERCISE I

1. What numbers represent the points P_1, P_2, P_3, A, B, in Fig. 2?

2. What are the values of P_2P_3, P_1P_2, P_3P_2, BP_3, P_3B?

3. If A be taken as origin, what are the numbers that represent P_1, P_2, O, P_3, B?

4. If the origin be moved two units to the right, how are the numbers representing different points affected? How if the origin be moved h units to the right? to the left?

4. Change of sign of a segment. Since any segment AB of the line contains the same number of units as BA, but is measured in the opposite direction, it follows that

$$BA = -AB, \text{ or } BA + AB = 0.$$

5. Addition of segments. Let A, B, and C be any three points on the line. Then

$$AC = AB + BC.$$

PROOF. Three cases arise:

(1) B between A and C, (2) A between B and C, (3) C between A and B.

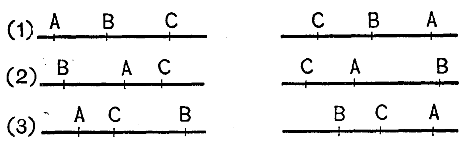


FIG. 3.

In (1), $AC = AB + BC$;

in (2), $AC = BC - BA = BC + AB$, by Art. 4,

or $AC = AB + BC$;

in (3), $AC = AB - CB = AB + BC$.

6. Subtraction of segments. By writing $-CB$ instead of BC in the equation of Art. 5, namely,

$$AC = AB + BC,$$

that equation becomes

$$\boldsymbol{AC = AB - CB.}$$

The results found in this and the preceding articles lead to the rules for geometric addition and subtraction of numbers that follow.

7. Geometric addition of numbers. Let P_1 and P_2 be two points on the line represented by the numbers x_1 and x_2 respectively. Then $OP_1 = x_1$, $OP_2 = x_2$.

Three cases arise:

(1) both numbers positive; (2) one number, say x_1, negative, the other positive; (3) both numbers negative.

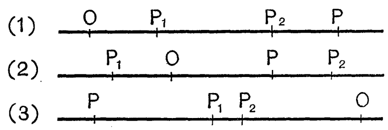


FIG. 4.

To represent geometrically the sum of x_1 and x_2 lay off from the end of x_1 a segment, P_1P, equal to x_2 and measured from P_1 in the same direction as x_2 is measured from O. Then

$$OP = x_1 + x_2.$$

For, in each case,

$$\begin{aligned} OP &= OP_1 + P_1P, \text{ by Art. 5,} \\ &= OP_1 + OP_2 \\ &= x_1 + x_2. \end{aligned}$$

8. Geometric subtraction of numbers. Consider again the three cases of Art. 7. To represent geometrically the difference $x_1 - x_2$ lay off from the end of x_1 a segment P_1P equal to $-x_2$, *i.e.* having the same numerical value as x_2 but opposite in direction.

Then $$OP = x_1 - x_2.$$

For, in each case,

$$OP = OP_1 + P_1P = OP - PP_1 = OP_1 - OP_2 = x_1 - x_2.$$

Another, and more important, expression of the difference is as follows:

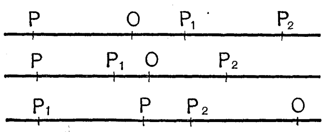


FIG. 5.

$x_1 - x_2 = OP_1 - OP_2 = OP_1 + P_2O$
$= P_2O + OP_1 = P_2P_1$, by Art. 5,
or $\boldsymbol{P_1P_2 = x_2 - x_1}$.

Hence, *the value of any segment of the line is equal to the number that represents the end minus the number that represents the beginning of the segment.*

This principle will be of frequent use hereafter.

ILLUSTRATION. In Fig. 6, if P_1, P_2, P_3 are three points on

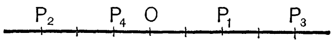


FIG. 6.

the line represented by the numbers 2, -3, 4 respectively, then

$$P_1P_2 = -3 - 2 = -5,\ P_2P_1 = 2 - (-3) = 5,$$
$$P_3P_2 = -3 - 4 = -7,\ P_1P_3 = 4 - 2 = 2,\ P_3P_1 = 2 - 4 = -2.$$

9. Relative position of points representing numbers. Let x_1 and x_2 be any two real numbers represented by the points P_1 and P_2 respectively.

By Art. 8, $$P_2P_1 = x_1 - x_2.$$

Now if $x_1 > x_2$ then $x_1 - x_2$ is positive, and conversely.

Therefore, if $x_1 > x_2$, P_2P_1 is positive, and hence P_1 lies to the right of P_2; if $x_1 < x_2$, P_2P_1 is negative, and hence P_1 lies to the left of P_2, and conversely.

Hence, *of the two points which represent two real numbers the point which represents the greater number lies farther to the right.*

E.g. in Fig. 6, P_1, which represents 2, lies to the right of P_2, which represents -3; P_4, which represents -1, lies to the

right of P_2, which represents -3. This agrees with the statement that 2 is greater than -3, and that -1 is greater than -3.

EXERCISE II

1. Represent geometrically the following pairs of numbers, their sum, the first minus the second, the second minus the first:

(*a*) 3, 2. (*b*) -2, 3. (*c*) 4, -3. (*d*) -5, -1.

2. In Fig. 5 express the following segments as the difference of the numbers representing the points: P_1P_2, P_2P_1, P_2O, OP_1, P_1O, OP_2.

3. In Fig. 5 what segments represent $x_1 - x_2$, $x_2 - x_1$, x_1, x_2, $-x_1$, $-x_2$?

4. In Fig. 6, by means of the principle in Art. 8, find the values of P_2P_1, P_3P_1, P_4O, P_4P_3, P_3P_4, OP_3, P_3O, P_3P_2.

II. COÖRDINATES OF POINTS IN THE PLANE

10. Location of a point. To determine the position of a point on a straight line one magnitude is sufficient; namely, the distance of the point, right or left, from a fixed point of the line. The number that represents a point on the line determines the position of the point when the origin is given.

In the plane, however, two magnitudes are necessary to determine the position of a point.

There are many ways of choosing these magnitudes. Two simple methods, and the only ones used in this book, are to consider the location of the point, (1) with reference to two intersecting straight lines, (2) with reference to a fixed line and a fixed point. A consideration of these two methods leads to the definitions of (1) **Cartesian Coördinates**, (2) **Polar Coördinates.**

11. Cartesian coördinates. Let two intersecting straight lines, OX and OY, be taken as lines of reference and an arbi-

trary length be chosen as a unit. Then to every point P in the plane there can be assigned a pair of real numbers as follows: Through the point P draw lines parallel to OX and OY, meeting OX and OY in M and N respectively. The pair of numbers which measure NP and MP is taken to represent the point P. To every position of P there corresponds one, and only one, pair of such numbers. In order that to every pair of real numbers there may correspond one, and only one, point, some agreement in regard to signs is necessary. To the agreement already made that a segment measured from left to right shall be positive, and one measured from right to left shall be negative, let there be added the agreement that a segment measured upward shall be positive, and a segment measured downward shall be negative. With this agreement in regard to signs there corresponds one, and only one, point in the plane to every pair of real numbers.

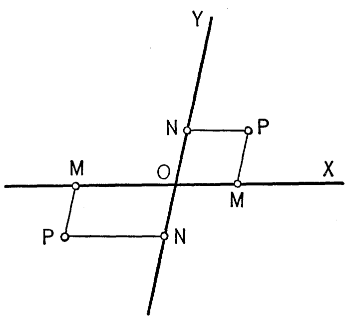


Fig. 7.

The lines OX and OY are called the **x-axis** and **y-axis** repectively.

The segments NP and MP are called respectively the **abscissa** and **ordinate** of P, and together are known as the Cartesian **coördinates of the point.**

It should be carefully noted that, from the definition, the abscissa of P is measured from the y-axis to P, and the ordinate of P is measured from the x-axis to P.

The abscissa and ordinate are most frequently denoted by x and y respectively, though other letters are sometimes used.

The point P is denoted by the coördinates inclosed in parentheses and separated by a comma, thus, (x, y) or $P(x, y)$.

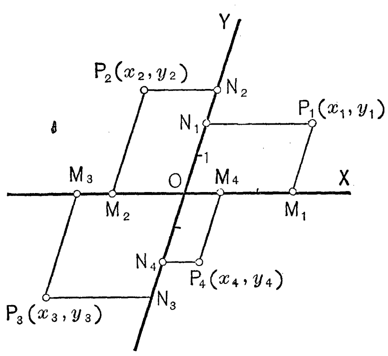


FIG. 8.

To distinguish one point from another, subscripts are often used. Thus in Fig. 8,

$$x_1 = N_1P_1 = OM_1 = 3,$$
$$y_1 = M_1P_1 = ON_1 = 2,$$
$$x_2 = N_2P_2 = OM_2 = -2,$$
$$y_2 = M_2P_2 = ON_2 = 3,$$
$$x_3 = N_3P_3 = -3,$$
$$y_3 = M_3P_3 = -3, \text{ etc.}$$

Also

$$M_3M_1 = M_3O + OM_1 = -x_3 + x_1,$$
$$M_1M_3 = M_1O + OM_3 = -x_1 + x_3,$$
$$M_4M_1 = M_4O + OM_1 = -x_4 + x_1, \text{ etc.}$$

EXERCISE III

1. Assume a pair of axes and locate the points (2, 3), (2, −3), (−2, 4), (−5, −6), (0, 2), (4, 0), (−1, 0), (0, −3), (0, 0).

2. In Fig. 8 express as the difference of two abscissas, M_2M_3, M_4M_2, M_3M_4, M_2M_1.

3. Express as the difference of two ordinates, N_2N_3, N_1N_4, N_3N_1, N_4N_2.

4. What segments represent $x_1 - x_2$, $x_3 - x_4$, $x_4 - x_1$, $x_3 - x_2$?

5. What segments represent $y_2 - y_1$, $y_4 - y_2$, $y_3 - y_2$, $y_1 - y_4$?

6. Where do all points lie that have the abscissa zero; that have the ordinate zero?

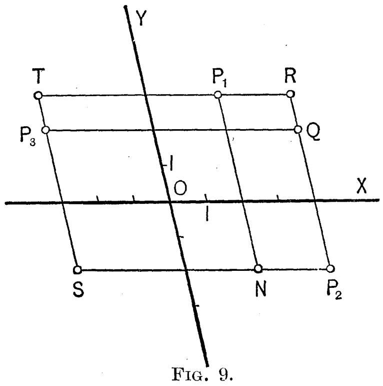


FIG. 9.

7. Where do all points lie that have the abscissa 2; that have the abscissa −3; that have the ordinate 2; that have the ordinate −4?

8. In Fig. 9 express as the difference of two abscissas, P_1R, P_2N, P_3Q, TR, SN; and as the difference of two ordinates, TP_3, P_1N, RQ, TS, P_2R, P_3S.

9. In Fig. 9 let P_1, P_2, P_3 have coördinates (2, 3), (4, -2), and (-3, 2), respectively, and find the values of SN, P_2S, RQ, RT, P_1N, SP_3, P_2R, QP_3.

12. Segments not parallel to an axis. Segments of lines not parallel to one of the coördinate axes will not have definite signs given to them. They will generally be considered as positive lengths, but where the two opposite directions along the same straight line are considered, one of them will be counted as opposite in sign to the other.

13. Rectangular coördinates. If the axes in the Cartesian coördinate system are at right angles to each other, the system is called the **rectangular system of coördinates.**

This system possesses the advantage of simplicity, in many problems, over that of oblique axes, and as most of the properties and relations of figures to be studied do not depend upon the system of coördinates used, the rectangular system will be used except where otherwise indicated.

14. Polar coördinates. The position of a point in the plane may be determined by the length of the line joining it to a fixed point, and the angle which this line makes with a fixed direction.

In Fig. 10 let O be a fixed point and OA a fixed line. Let P be any point in the plane. Then the segment OP and the angle AOP determine the location of P.

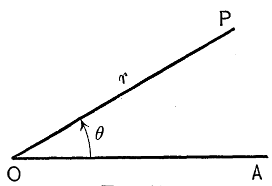


Fig. 10.

The segment OP is called the **radius vector**, and the angle AOP the **vectorial angle** of P.

Together they are known as the **polar coördinates** of P. They are usually denoted by r and θ, respectively, and the point indicated by (r, θ), or $P(r, \theta)$.

The fixed point O is called the **origin**, or **pole**; the fixed line OA the **initial line**, or **axis**.

The line OP is called the **terminal line** of the angle AOP.

With these definitions it is easy to see that any point in the plane may be represented by polar coördinates, both of which are positive, and with the angle less than 360°.

In order, however, to represent both positive and negative numbers by points, the following agreement in regard to signs is made: Positive angles will be measured in the counter-clockwise direction from the initial line; negative angles in the opposite direction. By a negative radius vector will be meant one laid off on the terminal line of the vectorial angle produced back through the pole.

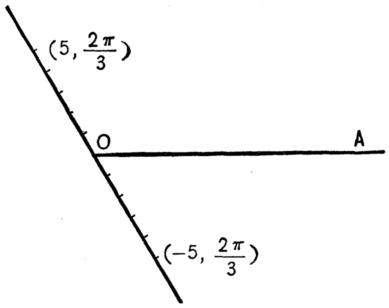


FIG. 11.

Thus, the points $\left(5, \frac{2\pi}{3}\right)$ and $\left(-5, \frac{2\pi}{3}\right)$ would be as indicated in Fig. 11.

With the above agreement in regard to signs it follows that to every pair of coördinates there is just one point in the plane, but to every point in the plane there corresponds an indefinite number of pairs of coördinates.

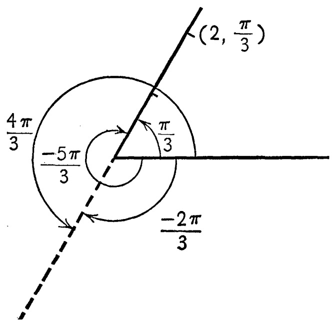


FIG. 12.

E.g. the point $\left(2, \frac{\pi}{3}\right)$ may also be represented by $\left(-2, -\frac{2\pi}{3}\right)$, $\left(-2, \frac{4\pi}{3}\right)$, $\left(2, -\frac{5\pi}{3}\right)$, or by

any other pair of coördinates obtained by increasing the angle of any of the above pairs by an integral multiple of 2π, the radius vector being unchanged.

If θ is restricted to being numerically less than 2π, the four pairs of values written above are the only ones that represent the given point.

NOTE. The student should remember that the unit of circular measure of an angle is the angle subtended at the center of a circle by an arc equal in length to a radius of the circle. This unit is called the radian.

From the definition it follows that π radians $= 180°$, where $\pi = 3.14159 \cdots$.

When an angle is represented by a letter or figure without the degree sign (°), it will be understood that the unit of measure is the radian.

EXERCISE IV

1. Plot in polar coördinates $(2, -30°)$, $\left(-4, \frac{5\pi}{6}\right)$, $\left(7, -\frac{2\pi}{3}\right)$, (π, π), $(\pi, \pi°)$, $(3, 2)$.

2. Plot in rectangular coördinates $(-3, 4)$, $(0, -3)$, $(0, 0)$, $(a, 0)$, $(0, b)$, $\left(\pi, -\frac{2\pi}{3}\right)$, $\left(6, \frac{\pi}{2}\right)$.

III. THE TRIGONOMETRIC FUNCTIONS

15. Definitions of the trigonometric functions. Having given any angle, assume a system of rectangular coördinates and place the vertex of the angle at the origin, with the initial line coinciding with the positive part of the x-axis; positive angles to be reckoned counter-clockwise and negative angles clockwise.

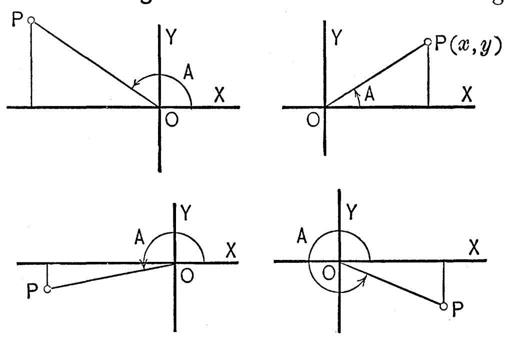


FIG. 13.

Assume any point P on the terminal line; let its coördinates be x and y, and its distance from the origin be r, counted always positive. Then, whatever the size of the angle, the following definitions are given:

$$\begin{aligned}
\text{sine of } A &= \text{ordinate/distance} = y/r,\\
\text{cosine of } A &= \text{abscissa/distance} = x/r,\\
\text{tangent of } A &= \text{ordinate/abscissa} = y/x,\\
\text{cotangent of } A &= \text{abscissa/ordinate} = x/y,\\
\text{secant of } A &= \text{distance/abscissa} = r/x,\\
\text{cosecant of } A &= \text{distance/ordinate} = r/y.
\end{aligned}$$

16. Formulas and tables. A set of the more important formulas connecting the trigonometric functions of angles, and a table of sines, cosines, and tangents are given at the back of the book.

17. The inverse trigonometric functions. The symbol $\sin^{-1} x$, read "anti-sine x," is used as equivalent to the words, "an angle whose sine is x."

Thus one value of $\sin^{-1}(\frac{1}{2})$ is $\frac{\pi}{6}$, or 30°; another value is $\frac{5\pi}{6}$.

In like manner the symbols $\cos^{-1} x$, $\tan^{-1} x$, etc., are used as equivalent to "an angle whose cosine is x," "an angle whose tangent is x," etc.

EXERCISE V

1. Find by the use of the table the sine, cosine, and tangent of each of the following angles: 20°, 17° 20′, 185°, 109° 40′, 290°, 165°, .2 radian, .72 radian, $\left(\frac{\pi}{5}\right)$ radian.

2. Given $A = \sin^{-1} .6$, find a value of A in the first quadrant, and one in the second quadrant.

3. Given $A = \tan^{-1} .4563$, find two values of A.

4. Find $\sin^{-1}(\tan 25°)$, $\sin(\tan^{-1} 3.26)$, $\sin(\sin^{-1} .35)$.

5. Show that $\sin(\sin^{-1} .5) = .5$, and that $\sin^{-1}(\sin 30°) = 30°$, or 150°, or 390°, etc.

18. Relation between rectangular and polar coördinates. Let the origin in the two systems be the same, and let the initial line coincide with the positive part of the x-axis.

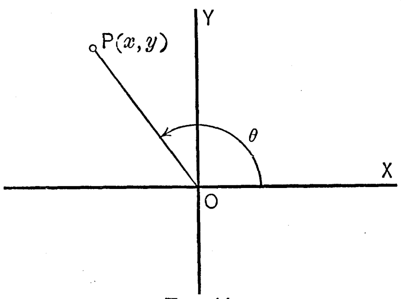


FIG. 14.

Let P be any point in the plane with rectangular coördinates x and y and polar coördinates r and θ (Fig. 14), the polar coördinates being so chosen that $\theta = \angle XOP$ and $r = OP$, where OP is positive.

Then from the definition of sine and cosine,

$$\frac{x}{r} = \cos\theta,\ \frac{y}{r} = \sin\theta,$$

or

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{r \cos \theta}, \\ \boldsymbol{y} &= \boldsymbol{r \sin \theta}. \end{aligned} \tag{1}$$

These equations express x and y in terms of r and θ. From the figure, or from these equations, r and θ can be expressed in terms of x and y. The resulting equations are

$$\begin{aligned} \boldsymbol{r} &= \sqrt{\boldsymbol{x^2 + y^2}}, \\ \boldsymbol{\theta} &= \boldsymbol{\tan^{-1}\left(\frac{y}{x}\right)}. \end{aligned} \tag{2}$$

EXERCISE VI

1. Show how to obtain eqs. (2) of Art. (18) from eqs. (1).

2. Show that if the polar coördinates of P be chosen so that θ differs from $\angle XOP$ by $180°$, and r is the negative of OP, eqs. (1) still hold.

3. Find the polar coördinates of the points whose rectangular coördinates are $(3, -7)$, $(4, 3)$, $(-2, 1)$, $(-4, -2)$.

4. Find the rectangular coordinates of the points whose polar coördinates are $(2, 30°)$, $(-3, 45°)$, $(4, -60°)$, $(-2, -15°)$.

5. In rectangular coördinates where do all points lie whose abscissas are zero; whose ordinates are zero; whose abscissas equal any constant

C; whose abscissas equal their ordinates; whose abscissas equal the negative of their ordinates?

6. What is true of the polar coördinates of points which satisfy each of the conditions of example 5?

7. In polar coördinates where do all points lie whose vectorial angles are zero; whose vectorial angles equal $30°$; whose vectorial angles equal any constant; whose radii vectores equal 5; whose radii vectores equal any constant C?

8. What equation is true of the rectangular coördinates of the points which satisfy each of the conditions in example 7?

9. Find the polar coördinates of the point whose rectangular coördinates are $(3.26, -2.67)$.

10. Find the rectangular coördinates of the point whose polar coördinates are $(6.34, 34° 16')$.

CHAPTER II

PROJECTIONS. LENGTHS AND SLOPES OF LINES. AREAS OF POLYGONS

I. PROJECTIONS

19. Projections by parallel lines. Through the beginning and end of a segment AB let lines parallel to a given direction be drawn to intersect a given line MN in C and D respectively. Then CD is called the **projection** of AB on MN, for the given direction.

The beginning and end of the projection are to be read in the same order as the beginning and end of the segment.

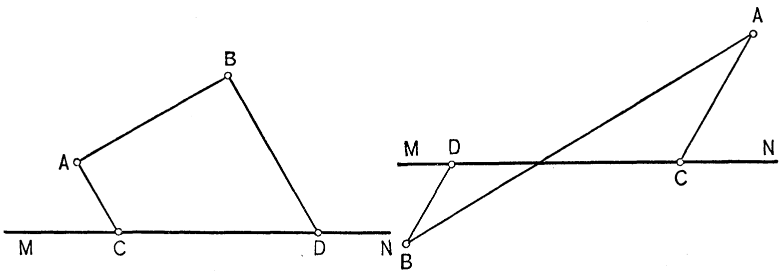


FIG. 15.

Thus CD is the projection of AB, while DC is the projection of BA. (Fig. 15.)

The direction, parallel to which the lines AC and BD are drawn, is called the direction of projection.

Evidently, the value of the projection depends upon, (1) the length of the segment, (2) the difference in direction of the segment and the line on which it is projected, and (3) upon the direction of projection. It is evident, also, that the pro-

jections of a given segment on parallel lines are equal, if the direction of projection is the same.

20. Orthogonal projection. If the direction of projection is perpendicular to the line on which the segment is projected, the projection is called **orthogonal.**

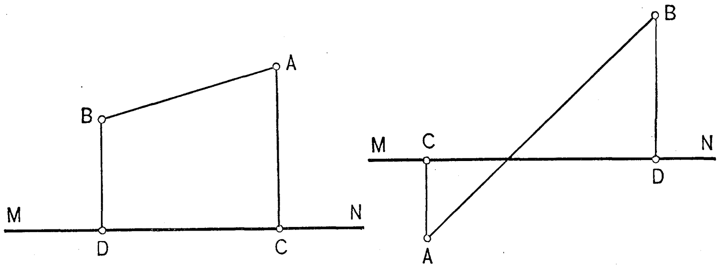


FIG. 16.

Thus in Fig. 16 CD is the orthogonal projection of AB on MN.

21. Projection in the direction of one coördinate axis on a line parallel to the other axis.

DEFINITION. The projection in the direction of the y-axis of a segment on a line parallel to the x-axis will be called the ***x*-projection** of the segment.

A similar definition is given for the ***y*-projection** of the segment.

Consider now the x-projection of any segment P_1P_2.

Let the coördinates of P_1 and P_2 be (x_1, y_1) and (x_2, y_2) respectively. Three cases may arise: P_1P_2 may lie wholly to the right of the y-axis, may cut the y-axis, or may lie wholly to the left of the y-axis. (Fig. 17.)

Let the projection in either case be M_1M_2, and let the line on which P_1P_2 is projected meet the y-axis at N. Then, in either case,

$$M_1M_2 = M_1N + NM_2 = -x_1 + x_2 = x_2 - x_1.$$

Therefore, *the x-projection of a segment is equal to the abscissa of the end of the segment minus the abscissa of the beginning.*

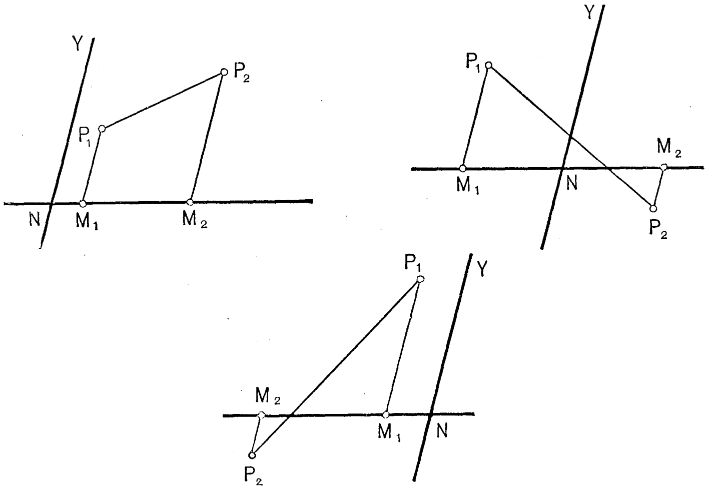


FIG. 17.

In like manner it can be shown that *the y-projection of a segment is equal to the ordinate of the end minus the ordinate of the beginning.*

EXAMPLE. The x-projection of the segment from $P_1(-1, 3)$ to $P_2(3, 2)$ is $3-(-1)=4$, and the y-projection is $2-3=-1$.

EXERCISE VII

1. Prove that the y-projection of a segment is equal to the ordinate of the end of the segment minus the ordinate of the beginning.

2. Find the x- and y-projections of the segments from the first to the second of each of the following pairs of points: $(2, 3)$, $(-2, 6)$; $(-3, -1)$, $(4, -5)$; $(1, -2)$, $(3, 7)$; (a, b), (c, d); $(0, 1)$, $(-2, 0)$; $(0, 0)$, $(3, -5)$; (u, v), (s, t).

Check the results by drawing the figure in each case.

3. If the axes are at right angles to each other, find the distance from the origin to $(3, 7)$; from the origin to (x, y).

4. If the axes are at right angles to each other, find the distance between $(-5, 3)$ and $(2, -6)$.

5. If the axes are rectangular, show that the distance between (x_1, y_1) and (x_2, y_2) is
$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

6. In rectangular coördinates the point (x, y) moves so as to keep at the distance 5 from the origin. Express this by means of an equation. What is the locus of the point?

7. What is the x-projection of a segment parallel to the y-axis; the y-projection of a segment parallel to the x-axis?

8. The vertices of a triangle are A, B, and C. Show that the sum of the projections of AB, BC, and CA on any line is zero, and that the projection of AC = the projection of AB + the projection of BC.

9. Show that the sum of the projections of the sides of any closed polygon taken in order, *i.e.* so that the beginning of each side is the end of the preceding, on any line is zero.

10. Show that if the sum of the projections of the sides of a polygon taken in order on one straight line is zero, the polygon is not necessarily closed; but if the sum of the projections taken in order on two non-parallel lines is zero, the polygon is closed.

II. LENGTHS AND SLOPES OF SEGMENTS. DIVISION OF SEGMENTS

22. Distance between two points. Numerical examples.

Example 1. To find the distance between the two points whose Cartesian coördinates are $(2, -4)$ and $(-3, 5)$, the angle between the axes being 60°.

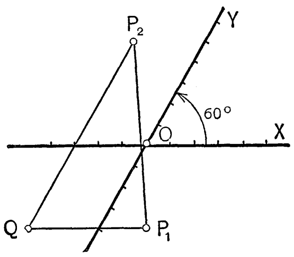


Fig. 18.

Let $(2, -4)$ be P_1, and $(-3, 5)$ be P_2.

Through P_1 and P_2 draw lines parallel, respectively, to the x- and y-axes, intersecting in Q. (Fig. 18.)

By the law of cosines from trigonometry,

$$\overline{P_1P_2}^2 = \overline{QP_1}^2 + \overline{QP_2}^2 - 2\,\overline{QP_1}\cdot\overline{QP_2}\cos P_1QP_2.$$

Here $\qquad QP_1 = 2 - (-3) = 5$, by Art. 21,

$$QP_2 = 5 - (-4) = 9,$$

$$\cos P_1QP_2 = \cos 60^\circ = \tfrac{1}{2}.$$

$$\therefore P_1P_2 = \sqrt{61} = 7.81 \cdots.$$

EXAMPLE 2. To find the distance between the points whose polar coördinates are $\left(2, \frac{2\pi}{3}\right)$ and $\left(5, -\frac{\pi}{6}\right)$.

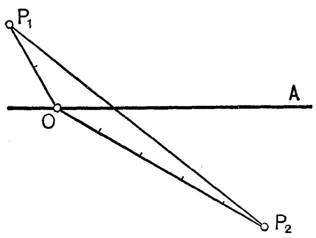


FIG. 19.

Let $\qquad \left(2, \frac{2\pi}{3}\right)$ be P_1, $\left(5, -\frac{\pi}{6}\right)$ be P_2,

and let $P_1P_2 = d$. (Fig. 19.)

By trigonometry,

$$\begin{aligned} d^2 &= \overline{OP}_1^2 + \overline{OP}_2^2 - 2\,\overline{OP}_1 \cdot \overline{OP}_2 \cos P_1OP_2 \\ &= 4 + 25 - 2 \cdot 2 \cdot 5 \cos \frac{5\pi}{6} \\ &= 4 + 25 + 20 \cos \frac{\pi}{6} \\ &= 29 + 17.32 \cdots. \end{aligned}$$

$$\therefore d = \sqrt{46.32} = 6.81 \text{ nearly.}$$

EXERCISE VIII

1. If the angle between the axes is 45°, find the distance between the points $(-3, 5)$ and $(4, 1)$.

2. If the angle between the axes is 80°, find the distance between $(6, 2)$ and $(-3, -4)$.

3. If the axes are rectangular, find the distance between (a, b) and (c, d).

4. Find the distance between the points whose polar coördinates are (6, 20°) and (4, $2^{(r)}$), where $2^{(r)}$ means 2 radians.

5. In polar coördinates find the distance between $\left(-3, \frac{\pi}{6}\right)$ and $\left(4, \frac{2\pi}{3}\right)$.

23. Distance between two points. General formula in rectangular coördinates.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points in rectangular coördinates, and let $d = P_1P_2$. Through P_1 and P_2 draw lines parallel, respectively, to the x- and y-axes to intersect in M.

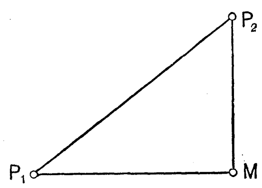


Fig. 20.

Then $d = \sqrt{\overline{P_1M}^2 + \overline{MP_2}^2}$.

But $P_1M = x_2 - x_1$, $MP_2 = y_2 - y_1$.

$\therefore d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

24. Distance between two points. General formula in polar coördinates.

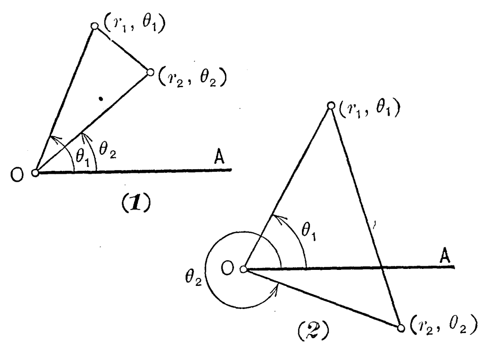


Fig. 21.

Let the two points be (r_1, θ_1) and (r_2, θ_2), and let the distance between them be d. (Fig. 21.)

There are two cases to consider: according as the difference between the vectorial angles is less than or greater than 180°.

In the first case

$$d^2 = r_1^2 + r_2^2 - 2\,r_1r_2 \cos(\theta_1 - \theta_2),$$

and in the second case

$$d^2 = r_1^2 + r_2^2 - 2\,r_1r_2 \cos[360° - (\theta_2 - \theta_1)].$$

These reduce to the one form

$$d = \sqrt{r_1^2 + r_2^2 - 2\,r_1r_2 \cos(\theta_1 - \theta_2)}.$$

EXERCISE IX

1. Find the distance between $(-4, 1)$ and $(3, 5)$, in rectangular coördinates.

2. Find the distance between $(3, 2)$ and $(-4, -5)$, in rectangular coördinates.

3. From a certain point O three other points, A, B, and C, are located as follows: A lies 3 mi. N. and $2\frac{1}{2}$ mi. E. from O, B lies 4 mi. S. and $1\frac{1}{2}$ mi. E. from O, and C lies 5 mi. W. and $1\frac{1}{4}$ mi. N. from O. Find the distances between the points A, B, and C, and the distance of each of the points from O correct to hundredths of a mile.

4. Find the distance between the points whose polar coördinates are $(4, 24°)$ and $(-2, 40°)$.

5. Find the lengths of the sides of the triangle whose vertices are $(5, -2)$, $(-4, 7)$, and $(7, -3)$, in rectangular coördinates.

6. Find the lengths of the sides of the triangle whose vertices are $(-2, 30°)$, $(4, 25°)$, and $(5, 115°)$.

25. The angle which one line makes with another.

DEFINITION. The angle which one line, L_1, makes with another, L_2, is the angle, not greater than 180°, measured counter-clockwise from L_2 to L_1.

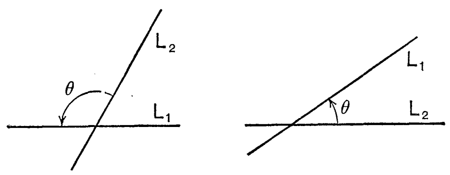


FIG. 22.

Thus, in Fig. 22, θ is the angle which L_1

makes with L_2. The supplement of θ is the angle which L_2 makes with L_1.

26. Inclination and slope of a line. The angle which a line makes with the x-axis, or with any line parallel to the x-axis, is called the **inclination** of the line.

This angle is to be measured from the positive direction of the x-axis toward the positive direction of the y-axis.

In rectangular coördinates, the **slope**, or **gradient**, of a line is the ratio of the change of the ordinate to the corresponding change of the abscissa of a point moving along the line. It is counted positive if the ordinate increases as the abscissa increases; negative if the ordinate decreases as the abscissa increases.

Thus, if, as a point moves along a line, the ordinate increases one unit to an increase of 3 units in the abscissa, the line has a slope of $\frac{1}{3}$; while if the ordinate decreases 1 unit to an increase of 3 units in the abscissa, the line has a slope of $-\frac{1}{3}$.

The inclinations of these lines are, respectively,

$$\theta = \tan^{-1}\tfrac{1}{3} \qquad = 18^\circ\, 26',$$

and $$\theta' = \tan^{-1}(-\tfrac{1}{3}) = 161^\circ\, 34'. \quad \text{(Fig. 23.)}$$

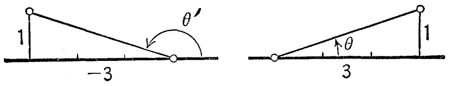


FIG. 23.

From the definitions of inclination and slope it follows that

slope = tangent of inclination,

or, designating the inclination of a line by θ and its slope by m,

$$m = \tan \theta.$$

If the axes are not rectangular, the equation,

slope = tangent of inclination,

is taken as definition of the slope.

27. Slope of a line through two points in terms of the rectangular coördinates of the points.

Let the two points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Through P_1 and P_2 draw lines parallel to the coördinate axes to meet in M. (Fig. 24.) Then whether the slope is positive or negative its value is given by the formula

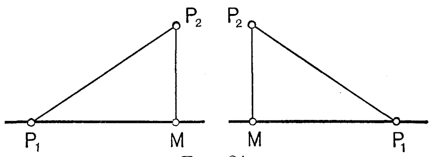


FIG. 24.

$$\textbf{slope} = \frac{MP_2}{P_1M} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If P_1 is the higher point, then slope $= \dfrac{y_1 - y_2}{x_1 - x_2}$, which is the same as the above.

Therefore, *in rectangular coördinates, the slope of a line through two points is the difference of the ordinates of the points divided by the corresponding difference of the abscissas of the points.*

28. Point dividing a line in a given ratio.*

EXAMPLE. To find the point which divides the line from $(-1, 5)$ to $(6, -4)$ in the ratio $3:2$.

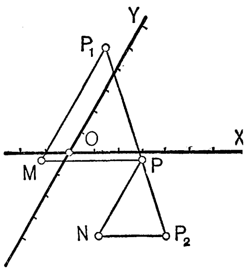


FIG. 25.

Let $(-1, 5)$ be P_1, $(6, -4)$ be P_2, and let the required point be $P(x, y)$. Then, by hypothesis,

$$\frac{P_1P}{PP_2} = \frac{3}{2}.$$

Through P, P_1, and P_2, draw lines parallel to the axes as in Fig. 25.

Then, from similar triangles,

$$\frac{MP}{NP_2} = \frac{P_1P}{PP_2} = \frac{3}{2},$$

* In this article and in several following articles the word "line" is frequently used in the sense of "segment of a line," where there is no doubt of the meaning.

and $$\frac{MP_1}{NP}=\frac{P_1P}{PP_2}=\frac{3}{2};$$

i.e. $$\frac{x+1}{6-x}=\frac{3}{2},$$

and $$\frac{5-y}{y+4}=\frac{3}{2},$$

from which $$x=3\tfrac{1}{5},\ y=-\tfrac{2}{5}.$$

Hence the required point is $(3\frac{1}{5}, -\frac{2}{5})$.

29. External division. The point P is said to divide the line P_1P_2 externally when it lies on the line produced. (Fig. 26.)

The segments into which P divides P_1P_2 are defined to be P_1P and PP_2. The first segment is that from the beginning of the line to the point of division, and the second segment is that from the point of division to the end of the line. Since these segments are measured in opposite directions, they are opposite in sign. Hence their ratio is negative. The first and second segments must correspond respectively to the first and second terms of the given ratio into which P is to divide P_1P_2.

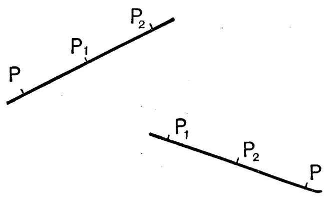


Fig. 26.

30. Example of external division.

To find the point which divides the line from $(-1, 5)$ to $(4, 7)$ in the ratio $-\frac{2}{3}$.

Let $(-1, 5)$ be P_1, $(4, 7)$ be P_2, and let the required point be $P(x, y)$.

Then $$\frac{P_1P}{PP_2}=-\frac{2}{3}.$$

Since P_1P must be numerically less than PP_2, P must lie nearer to P_1 than to P_2, *i.e.* P must lie on the portion of the line extended through P_1.

Project the segments so as to obtain their x- and y-projections. (Fig. 27.)

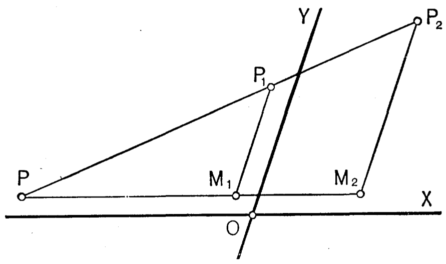


FIG. 27.

Then $$\frac{M_1P}{PM_2}=\frac{P_1P}{PP_2}=-\frac{2}{3},$$

and $$\frac{M_1P_1}{P_2M_2}=\frac{P_1P}{PP_2}=-\frac{2}{3}.$$

$\therefore$ $$\frac{x+1}{4-x}=-\frac{2}{3},$$

and $$\frac{5-y}{y-7}=-\frac{2}{3},$$

from which $$x=-11,\ y=1.$$

Hence the required point is $(-11, 1)$.

EXERCISE X

1. Find the point which divides the line from $(-3, 1)$ to $(6, -5)$ in the ratio $-\frac{5}{2}$. *Ans.* $(12, -9)$.

2. Show that the point which bisects the line joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

3. Find the ratio in which the line from $(2, 0)$ to $(6, 0)$ is divided by $(1, 0)$; by $(5, 0)$; by $(9, 0)$.

4. The point $P(2, k)$ is on the line joining $P_1(-2, 3)$ and $P_2(4, -7)$; find the ratio into which P divides P_1P_2, and the value of k.

31. General formulas for a point dividing a line in a given ratio.

Let the line from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ be divided by $P(x, y)$ in the ratio $r:1$.

There are three cases to consider:

(1) P between P_1 and P_2,

(2) P on the line produced through P_1,

(3) P on the line produced through P_2.

In (1) r may have any positive value,

in (2) r is negative and numerically less than 1,

in (3) r is negative and numerically greater than 1.

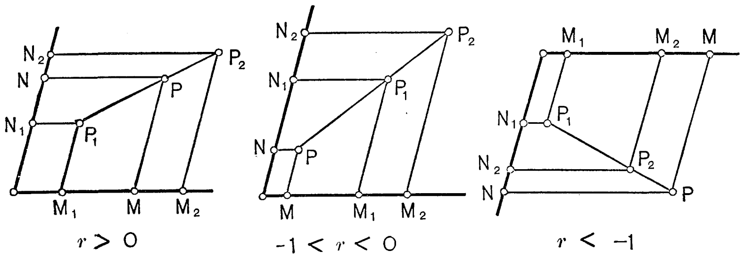


FIG. 28.

Project P_1P and PP_2 on any two lines parallel to the axes. (Fig. 28.) In either of the three cases,

$$\frac{M_1M}{MM_2} = \frac{P_1P}{PP_2} = r, \text{ and } \frac{N_1N}{NN_2} = \frac{P_1P}{PP_2} = r,$$

or
$$\frac{x - x_1}{x_2 - x} = r, \text{ and } \frac{y - y_1}{y_2 - y} = r,$$

from which
$$\boldsymbol{x = \frac{x_1 + rx_2}{1 + r}}, \qquad \boldsymbol{y = \frac{y_1 + ry_2}{1 + r}}.$$

EXERCISE XI

1. Find the point which divides the line from $(-1, 3)$ to $(6, -5)$ in the ratio $3:2$.

2. Find the point which divides the line from $(3, \frac{3}{2})$ to $(-5, 8)$ in the ratio $-\frac{4}{3}$; in the ratio $-\frac{3}{4}$.

3. Find the external point on the line joining $P_1(a, b)$ and $P_2(c, d)$ which is n times as far from P_1 as from P_2.

4. Find the points which trisect the line joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

5. The point P divides the line P_1P_2 in the ratio $r : 1$; trace the variation in r as P moves along the line internally from P_1 to P_2, then on from P_2 to ∞, and then, changing to the other side of P_1, comes in from $-\infty$ to P_1.

32. Angle between two lines of given slopes.

EXAMPLE. Let two lines L_1 and L_2 have slopes -2 and 3 respectively; to find the angle which L_1 makes with L_2.

Let L_1 and L_2 make angles θ_1 and θ_2 respectively with the x-axis, and let the angle which L_1 makes with L_2 be ϕ. Through the intersection of the lines draw a line parallel to the x-axis. (Fig. 29.) Then it is seen that

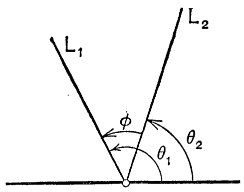


FIG. 29.

$$\phi = \theta_1 - \theta_2.$$

Hence
$$\tan \phi = \tan (\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}.$$

But
$$\tan \theta_1 = -2, \quad \tan \theta_2 = 3.$$

$$\therefore \tan \phi = \frac{-2-3}{1-2 \cdot 3} = 1.$$

$$\therefore \phi = 45^\circ.$$

33. The angle between two lines. General formula.

Let two lines, L_1 and L_2, have slopes m_1 and m_2 respectively; to find the angle which L_1 makes with L_2.

Let the angles which L_1 and L_2 make with the x-axis be θ_1 and θ_2 respectively. Then $m_1 = \tan \theta_1$, $m_2 = \tan \theta_2$.

Let ϕ be the angle which L_1 makes with L_2.

Through the intersection of L_1 and L_2 draw a line parallel to the x-axis. Then (Fig. 30),

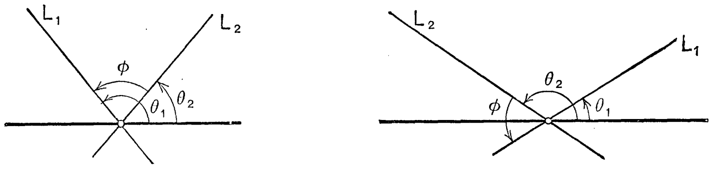


FIG. 30.

case (i), $\theta_1 > \theta_2$,

$$\phi = \theta_1 - \theta_2;$$

case (ii), $\theta_1 < \theta_2$,

$$\phi = 180° - (\theta_2 - \theta_1)$$
$$= 180° + (\theta_1 - \theta_2),$$

and in either case

$$\tan \phi = \tan(\theta_1 - \theta_2)$$
$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$
$$= \frac{m_1 - m_2}{1 + m_1 m_2}.$$

34. Condition for parallel lines, and for perpendicular lines. If the two lines of the preceding article are parallel, $\tan\theta_1 = \tan\theta_2$, and hence $m_1 = m_2$. If the two lines are perpendicular, $\tan\phi = \tan 90° = \infty$, and hence $1 + m_1 m_2 = 0$.

Conversely, if $m_1 = m_2$, $\tan\phi = 0$, $\therefore \phi = 0$, and therefore the lines are parallel.

If $1 + m_1 m_2 = 0$, $\tan\phi = \infty$, $\therefore \phi = 90°$, and therefore the lines are perpendicular.

Therefore, the condition that two lines of slopes m_1 and m_2 be parallel is $m_1 = m_2$; the condition that they be perpendicular is $1 + m_1 m_2 = 0$, or $m_1 = -\frac{1}{m_2}$.

EXERCISE XII

For rectangular axes. Draw a figure in each case.

1. Show that the line joining (3, 2) and (– 2, – 13) is perpendicular to the line joining (1, 3) and (4, 2).

2. Show that (– 1, – 2), (3, 2), and (– 3, 0) are the vertices of a right triangle. Find the other angles.

3. Where does a line cut the x-axis if it passes through (2, – 3) and is parallel to the line through (– 1, 5) and (4, – 2)?

4. A line is drawn perpendicular to the line through $P_1(-2, 5)$ and $P_2(4, -3)$ at its middle point; find a point P on this perpendicular whose abscissa is 3, and show that P is equidistant from P_1 and P_2.

5. The vertices of a triangle are (7, 4), (– 2, – 5), and (3, – 10); show that the line joining the middle points of two sides is parallel to the third side, and is half as long, by using formulas for slope and distance.

6. Find a fourth point which with the three given in example 5 form the vertices of a parallelogram.

7. Two lines, L_1 and L_2, make $\tan^{-1} 2$ and $\tan^{-1} - 4$ respectively with the x-axis; find the angle which L_1 makes with L_2.

8. The vertices of a triangle are $P_1(-1, 5)$, $P_2(3, -4)$, and $P_3(6, 2)$; find the slopes of the sides and the angle at P_1.

9. Show by their slopes that the line joining (– 3, 4) and (6, 1) is parallel to the line joining (7, 2) and $(5, \frac{8}{3})$.

10. A line L makes an angle of 45° with the line through (1, 1) and (6, 8); find the slope of L and the angle which it makes with the x-axis.

11. L_1 passes through (4, 5) and (6, – 3). L_2 is perpendicular to L_1; find the slopes of L_1 and L_2.

12. L_1 has a slope m. The angle which L_2 makes with the x-axis is double the angle which L_1 makes with the x-axis; what is the slope of L_2?

13. The slope of one line is 3.728 and of another – .324; find the acute angle between them.

14. Find the slope of a line which makes an angle of – 42° with a line of slope .4364.

15. A line passes through (6, – 3) and has a slope .324; find a point on the line with abscissa 1.2.

16. A line cuts the x-axis at $(a, 0)$ and makes $\tan^{-1} m$ with the x-axis; find where it cuts the y-axis.

17. A line passes through $(a, 0)$ and makes $\tan^{-1} m$ with a line of slope n; find its slope, and where it cuts the y-axis.

III. AREAS OF POLYGONS

35. Area of a triangle in terms of the coördinates of its vertices.

EXAMPLE 1. To find the area of a triangle whose vertices in rectangular coördinates are $P_1(-2, 3)$, $P_2(4, -1)$, and $P_3(1, -6)$.

Through the lowest vertex, P_3 (Fig. 31), draw a line parallel to the x-axis, and from the other vertices drop perpendiculars to this line, meeting it in M_1 and M_2.

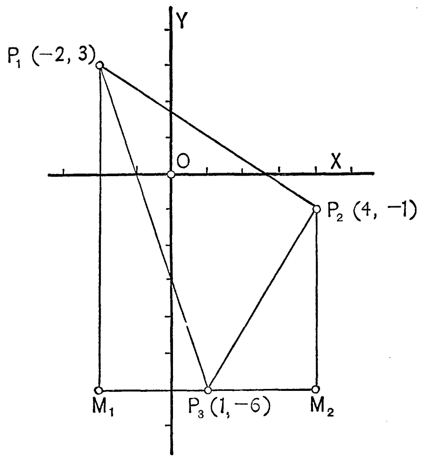


FIG. 31.

Then the area required is equal to

$$\begin{aligned}
&\text{area of } M_2P_2P_1M_1 \\
&\quad - \text{area of } P_3M_2P_2 \\
&\quad - \text{area of } M_1P_3P_1 \\
&= \tfrac{1}{2}\, M_1M_2(M_1P_1 + M_2P_2) \\
&\quad - \tfrac{1}{2}\, P_3M_2 \cdot M_2P_2 \\
&\quad - \tfrac{1}{2} M_1P_3 \cdot M_1P_1 \\
&= \tfrac{1}{2} \cdot 6\,(9+5) \\
&\quad - \tfrac{1}{2} \cdot 3 \cdot 5 \\
&\quad - \tfrac{1}{2} \cdot 3 \cdot 9. \\
&= 21.
\end{aligned}$$

If $P_1P_2P_3$ represents a triangular field to a scale of 1 space $= n$ ft., then the area of the field is $21\ n^2$ sq. ft.

EXAMPLE 2. To find the area of the triangle whose vertices in polar coördinates are $(3, 60°)$, $(-2, 125°)$, and $(5, 215°)$.

The area required is the sum of the areas of the triangles OP_2P_1, OP_1P_3, OP_3P_2 (Fig. 32).

The area of a triangle is equal to one half the product of two sides and the sine of the included angle.

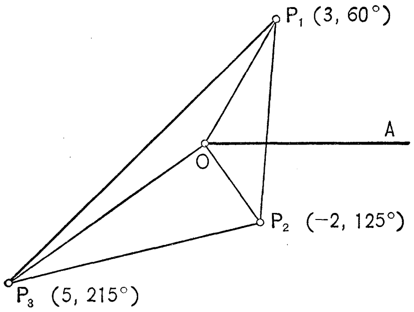


FIG. 32.

$\therefore$ the required area

$$= \tfrac{1}{2}\,OP_1 \cdot OP_3 \sin P_1OP_3 + \tfrac{1}{2}\,OP_3 \cdot OP_2 \sin P_3OP_2 + \tfrac{1}{2}\,OP_2 \cdot OP_1 \sin P_2OP_1$$
$$= \tfrac{1}{2} \cdot 3 \cdot 5 \sin 155^\circ + \tfrac{1}{2} \cdot 5 \cdot 2 \sin 90^\circ + \tfrac{1}{2} \cdot 2 \cdot 3 \sin 115^\circ$$
$$= \tfrac{1}{2}(15 \sin 25^\circ + 10 + 6 \cos 25^\circ) = 10.89.$$

EXERCISE XIII

1. Find the area of the triangle whose vertices in rectangular coördinates are $(3, -5)$, $(-8, 6)$, and $(9, 2)$.

2. Find the area of the triangle whose vertices in polar coördinates are $\left(5, \frac{\pi}{3}\right)$, $\left(-6, \frac{2\pi}{3}\right)$, and $\left(3, \frac{3\pi}{4}\right)$.

3. Find the area of a triangle whose vertices in rectangular coördinates are $(0, 0)$, (x_1, y_1), and (x_2, y_2).

4. Find the area of a triangle whose vertices in polar coördinates are $(0, 0)$, (r_1, θ_1), and (r_2, θ_2).

5. Find the area of the quadrilateral whose vertices in rectangular coördinates are $(-2, 5)$, $(7, 9)$, $(10, -3)$, and $(-6, -9)$.

36. Area of a triangle. General formula in rectangular coördinates. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ be the vertices of a triangle in rectangular coördinates; to find the area of the triangle.

Through the lowest vertex (P_2 in Fig. 33) draw a line parallel to the x-axis, and from the other vertices drop perpendiculars to this line, meeting it in M_1 and M_3. Then

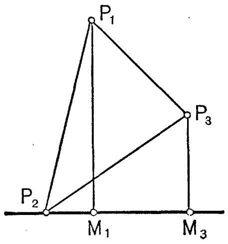


FIG. 33

area of triangle $P_1P_2P_3$

$$= \text{area of trapezoid } M_3P_3P_1M_1$$
$$+ \text{area of triangle } P_1P_2M_1$$
$$- \text{area of triangle } P_2M_3P_3$$
$$= \tfrac{1}{2}(M_1P_1 + M_3P_3)\cdot M_1M_3 + \tfrac{1}{2}\,P_2M_1\cdot M_1P_1 - \tfrac{1}{2}\,P_2M_3\cdot M_3P_3$$
$$= \tfrac{1}{2}\left[(y_1 - y_2 + y_3 - y_2)(x_3 - x_1) + (x_1 - x_2)(y_1 - y_2) - (x_3 - x_2)(y_3 - y_2)\right],$$

or, ***area*** $\boldsymbol{P_1P_2P_3 = \tfrac{1}{2}\,(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)}$.

This may be written in the determinant form

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

In Fig. 33 the succession of subscripts 1, 2, 3, is obtained by going around the triangle counter-clockwise. If the points had been so lettered that in following the above order it would be necessary to go around the triangle clockwise, the area would have been found to be minus the above expression.

This can be seen to be true by exchanging two of the subscripts, say 1 and 2, in Fig. 33, and making the same exchange in the formula. The change in the figure changes the order

from counter-clockwise to clockwise, and the change in the formula just changes the sign of the whole expression.

37. Area of a triangle. General formula in polar coördinates. Let $P_1(r_1, \theta_1)$, $P_2(r_2, \theta_2)$, and $P_3(r_3, \theta_3)$ be the vertices of a triangle in polar coördinates; to find the area of the triangle.

Two cases are to be distinguished, according as the pole lies without or within the triangle. The second case will occur only when the difference between the vectorial angles of two of the vertices is greater than 180°.

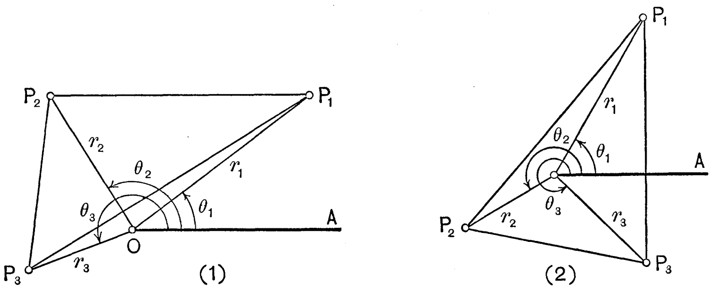


FIG. 34.

In case (1) the area of the triangle $P_1P_2P_3$ is equal to the area of triangle OP_1P_2 + area of triangle OP_2P_3 − area of triangle OP_1P_3

$$= \tfrac{1}{2} r_1r_2 \sin(\theta_2 - \theta_1) + \tfrac{1}{2} r_2r_3 \sin(\theta_3 - \theta_2) - \tfrac{1}{2} r_1r_3 \sin(\theta_3 - \theta_1)$$
$$= \tfrac{1}{2} [r_1r_2 \sin(\theta_2 - \theta_1) + r_2r_3 \sin(\theta_3 - \theta_2) + r_3r_1 \sin(\theta_1 - \theta_3)].$$

In case (2) the area of triangle OP_1P_3 must be added to the areas of the other two triangles, instead of subtracted from them, as in case (1); but area of OP_1P_3 is here equal to $\tfrac{1}{2} r_1r_3 \sin[360° - (\theta_3 - \theta_1)]$ which is equal to $-\tfrac{1}{2} r_1r_3 \sin(\theta_3 - \theta_1)$. The formula for the area of the triangle sought reduces therefore to the same as in case (1).

Just as in the case of the area in rectangular coördinates, the above formula would give the negative of the area if the sub-

scripts were so arranged that in following the order 1, 2, 3, it would be necessary to go around the triangle clockwise.

38. Area of a polygon. General formula in rectangular coördinates. If the origin be one of the vertices of a triangle whose other vertices are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the formula for the area of the triangle given in Art. 36 becomes

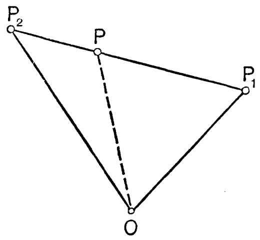


FIG. 35.

$$\tfrac{1}{2}(x_1y_2 - x_2y_1),$$

provided that in going around the triangle counter-clockwise the vertices are passed in the order P_1, P_2, and O. This area of the triangle OP_1P_2 may be thought of as generated by a line OP, initially in the position OP_1, turning counter-clockwise about O to the final position OP_2, the point P moving along the line P_1P_2. With this conception of the area, it must be noted that it is the abscissa, x_1, of the initial position, P_1, of P which comes first in the formula for the area, $\frac{1}{2}(x_1y_2 - x_2y_1)$.

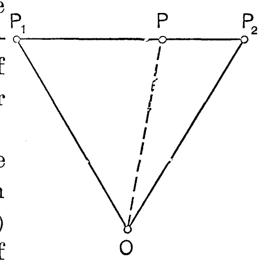


FIG. 36.

If the line OP must turn clockwise from the position OP_1 to the position OP_2, then the expression $\frac{1}{2}(x_1y_2 - x_2y_1)$ is equal to the negative of the area of the triangle OP_1P_2.

Let $\frac{1}{2}(x_1y_2 - x_2y_1)$ be denoted by Δ. Thus

$$\Delta = \tfrac{1}{2}(x_1y_2 - x_2y_1).$$

Consider now any polygon whose vertices in rectangular coördinates are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $\cdots$ $P_n(x_n, y_n)$, the vertices being so lettered that in going around the polygon counter-clockwise the vertices are passed in the order P_1, P_2, $\cdots$ P_n.

For definiteness let $n = 6$, and let the polygon be as shown in Fig. 37, the origin being outside of the polygon. Let a point P

start at P_1, traverse the perimeter of the polygon counter-clockwise, and return to P_1. The line OP generates in order the triangles OP_1P_2, OP_2P_3, $\cdots$ OP_6P_1. Now the area generated by OP which lies without the polygon is generated twice, with OP turning once clockwise, once counter-clockwise; or else is generated four times with OP turning twice clockwise, twice counter-clockwise; but the area within the polygon is generated once, with OP turning counter-clockwise; or else is generated three times, with OP turning once clockwise, twice counter-clockwise. Therefore if the expression Δ be formed for each of the triangles OP_1P_2, OP_2P_3, $\cdots$ OP_6P_1, and their sum taken, all the area generated by OP will be cancelled out except that within the polygon and that area will be counted just once. Therefore the area of the polygon is equal to

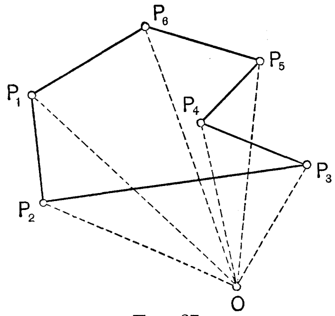


FIG. 37.

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_5 - x_5y_4 + x_5y_6 - x_6y_5 + x_6y_1 - x_1y_6).$$

A convenient method of arranging the coördinates for the computation of the area is as follows: *Write down in succession the abscissas of the vertices taken in order counter-clockwise around the polygon, repeating the first abscissa at the last; under the abscissas write the corresponding ordinates:*

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_1 \end{matrix}$$

Then multiply each abscissa by the following ordinate and take the sum of the terms obtained; multiply each ordinate by the following abscissa and take the sum of the terms obtained. The area is half of the first sum minus half of the second.

EXERCISE XIV

1. The vertices of a polygon taken in order are $(6, 1)$, $(9, -4)$, $(3, -10)$, $(-3, -5)$, $(-6, -8)$, $(-12, 0)$ and $(-4, 6)$; find the area of the polygon.

2. The distances north of a fixed east and west line of four points A, B, C, D are respectively 32.6 ft., 65.1 ft., 80.3 ft., 51.7 ft., and their distances east of a fixed north and south line are respectively 25.3 ft., 48.2 ft., 94.5 ft., 106 ft.; find the area of the quadrilateral $ABCD$.

3. The distances of four points A, B, C, D from a point O are respectively 120 ft., 216 ft., 320 ft., and 65 ft., and their directions from O are respectively E. 25° N., N. 32° W., S. 74° W., E. 67° S.; find the area of $ABCD$.

4. The vertices of a triangle are $(3, -2)$, $(-4, 1)$, and $(-8, -5)$; find (a) the area, (b) the lengths of the sides, (c) the slopes of the sides, (d) the angles.

5. Show (a) by the lengths of the sides, (b) by the slopes of the sides, that the quadrilateral whose vertices are $(1, 2)$, $(3, -2)$, $(-1, -3)$, and $(-3, 1)$ is a parallelogram. Find its area.

6. Show by means of the slopes of the lines that the line joining the middle points of two sides of any triangle is parallel to the third side. Show also that its length is half that of the third side.

7. The vertices of a triangle are P_1, P_2, P_3; find the point which divides the line from P_1 to the middle point of P_2P_3 in the ratio $2:1$. Show that, using either of the vertices in like manner, the same point is obtained, and hence that the three medians of a triangle meet in a point.

8. In the formula for the area of a triangle in rectangular coördinates, substitute the values of the rectangular coördinates in terms of the polar coördinates and obtain the formula for the area of the triangle in terms of polar coördinates.

9. The line joining (a, b) and (c, d) is divided into four equal parts; find the points of division.

10. Show analytically that the middle points of the sides of any quadrilateral are the vertices of a parallelogram.

11. Prove that the middle point of the line joining the middle points of two opposite sides of any quadrilateral has an abscissa equal to one fourth the sum of the abscissas of the vertices of the quadrilateral, and find the similar relation for the ordinates. What conclusion can you draw?

12. The point $(2, k)$ is equidistant from $(-5, 7)$ and $(3, 4)$; find k.

13. The point (x, y) is equidistant from $(2, -1)$ and $(7, 4)$; write the equation which x and y must satisfy. What is the locus of (x, y)?

14. Express by an equation the condition that the point (x, y) is distant 5 from $(2, 3)$. What is the locus of the point (x, y)?

15. Show that the line joining $(4, -4,)$ and $(-2, -1)$ is perpendicular to the line joining $(3, 1)$ and $(1, -3)$.

16. Find the angle which the line whose slope is 6.324 makes with the line whose slope is $-.657$.

17. Find the slope of a line which makes an angle of 30° with a line whose slope is 3.

18. The line L_1 makes an angle of 40° with the x-axis, and the line L_2 makes an angle whose tangent is 2 with L_1; find the slope of L_2.

19. If L_1 makes $\tan^{-1} a$ with the x-axis, and L_2 makes $\tan^{-1} b$ with L_1, find the slope of L_2.

20. The angle from L_1 clockwise to L_2 is $\tan^{-1}(\frac{3}{4})$, and the angle from L_2 counter-clockwise to the x-axis is $\tan^{-1}(-\frac{5}{7})$; find the slope of L_1.

CHAPTER III

GRAPHICAL REPRESENTATION OF A FUNCTION; EQUATION OF A LOCUS

39. Function and variable. One quantity is said to be a **function** of a second quantity when to every value of the second there corresponds one or more values of the first.

Thus in the equation $v = gt$, which expresses the velocity of a body falling freely in a vacuum in terms of the time, the velocity, v, is a function of the time, t.

Again, in the equation $pv =$ a constant, the formula which expresses the relation between the pressure and volume of a gas kept at constant temperature, either of the quantities p or v is a function of the other one.

The quantity which may take, or to which may be assigned, arbitrary values is called the **independent variable**, or often simply the **variable**, and a function of this variable is often called the **dependent variable.**

According to the above definition of a function any constant may be regarded as a function which takes the same value for all values of the variable.

I1 to every value of the variable there is just one value of the function, the function is said to be a single-valued function of the variable. If two, three, or more values of the function exist for every value of the variable, the function is called respectively a double-valued, triple-valued, or, in general, a multiple-valued function of the variable.

Thus in $v = 32\,t$, v is a single-valued function of t, and in $y^2 = 4\,x$, y is a double-valued function of x. On the other hand, x is a single-valued function of y, if y be taken as the independent variable.

40. The graph of a function. It is not always possible to express by means of an equation the value of a function in terms of the variable. When, however, there are known several pairs of corresponding values of two quantities, one of which depends upon the other, a graphical representation of one of the quantities as a function of the other may be made which will exhibit in an instructive way the dependence of one of the quantities upon the other.

To illustrate this consider the following examples.

EXAMPLE 1. It was found that when a certain rod of steel was subjected to tension, the values of the extension of the rod in terms of the tension were as shown in the following table, in which T is the number of pounds of tension per square inch of cross-section of the rod and ϵ is the number of units of extension per unit length of the rod, the initial tension being 1000 lb.

T	1000	5000	10,000	20,000	30,000	40,000	50,000	51,000
ϵ	0	.0003	.0009	.0019	.0030	.0040	.0053	.0056
T	52,000	54,000	56,000	58,000	60,000	70,000	80,000	
ϵ	.0058	.0064	.0075	.0089	.0113	.0272	.0500	

Take the values of ϵ as abscissas and the values of T as ordinates and plot the points representing the corresponding values of ϵ and T. Then draw a smooth curve through these points. On the assumption that as the tension changes gradually, passing through all values between the first and last values of the tension that are given, the extension also changes gradually, the smooth curve through the plotted points may be taken as a graphical representation of T as a function of ϵ in the sense that the coördinates of any point on the curve are corresponding values of ϵ and T.

In general the more points that are determined by known values of the variables the more accurately will the curve represent the function. Of course, too, these points should be somewhat evenly separated.

Outside the range of values given, no information can be drawn from the curve concerning the values of the function for a given value of the variable.

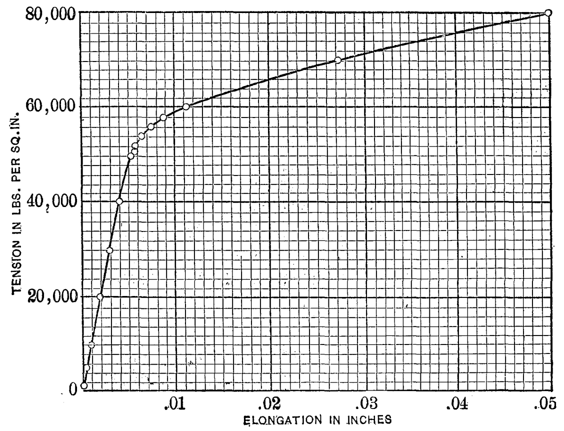


FIG. 38.

The curve does not give any information that is not contained in the table, but gives the same information in such a way as to bring out relations that are not readily observed from the table.

From the curve it is seen that as long as T is less than about 50,000 the extension is proportional to the tension, the points of the curve lying on a straight line approximately, but that when T passes through the value 50,000 the extension increases more and more rapidly as T increases.

Also the value of T corresponding to an assumed value of ϵ may be found approximately from the curve by measuring the value of the ordinate of the point of the curve which has the assumed value of ϵ as abscissa. Likewise the value of ϵ corresponding to an assumed value of T may be found.

EXAMPLE 2. The following table shows the number B of

beats per minute of a simple pendulum of length L centimeters for certain values of L:

L	10	12	15	20	25	30	40	50	60	70	80	90	100
B	190	172	154	136	120	110	95	85	78	72	67	63	60

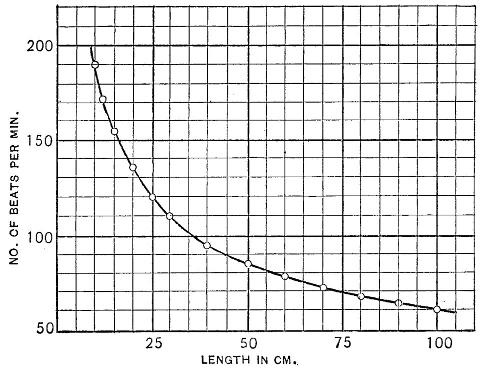


FIG. 39.

Take the values of L as abscissas and the values of B as ordinates and plot the points representing the corresponding values of L and B. The curve drawn through these points shows graphically the manner in which B depends upon L.

It also enables one to pick out approximately the value of B for a given value of L within the limits given, or the value of L for a given value of B.

41. Equation of a locus. In each of the two preceding examples a curve was drawn such that the coördinates of all of its points were corresponding values of the function and variable, but no equation was found which expressed the dependence of the function upon the variable.

In each of the examples to be next studied some simple locus of points will be considered, and the equation which expresses the dependence of the ordinate of any point of the

locus upon the abscissa of the point will be derived. This equation will be known as the **equation of the locus.**

DEFINITION. **The equation of a locus** is an equation between the coördinates of any point of the locus.

The locus, on the other hand, is called the **locus of the equation.**

42. Two fundamental problems. The two fundamental problems of Plane Analytic Geometry are:

(1) Having given a locus of points determined by certain geometric conditions, to find the equation of that locus.

(2) Having given an equation in two variables, to find by a study of the equation the form and properties of the locus which it represents.

In this chapter some examples illustrating the methods of finding the equation of a given locus will be considered, and in the next chapter some methods of obtaining the locus when the equation is given will be studied.

43. Illustrations. EXAMPLE 1. Consider the locus of a point which moves along the straight line passing through the points $P_1(3, -1)$ and $P_2(-5, 4)$. If any point $P(x, y)$ be taken on this line, the value of the ordinate clearly depends upon the value of the abscissa of the point. That is, y is a function of x.

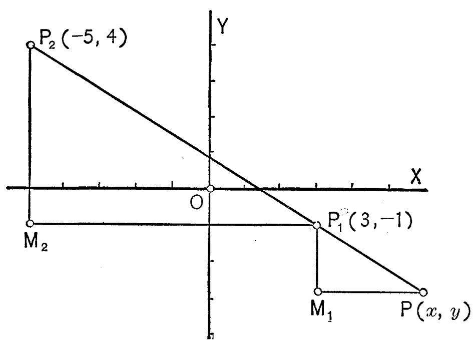


FIG. 40.

To find the law, or equation, which expresses the dependence of y upon x, draw through P, P_1, and P_2 lines parallel to the axes to form the triangles PM_1P_1 and $P_1M_2P_2$ as in Fig. 40.

Then by similar triangles

$$\frac{M_2P_1}{M_1P}=\frac{M_2P_2}{M_1P_1},$$

i.e.

$$\frac{3-(-5)}{x-3}=\frac{4-(-1)}{-1-y},$$

which reduces to $$5x+8y=7. \tag{1}$$

If $P(x, y)$ is a point not on the line through P_1 and P_2, the triangles PM_1P_1 and $P_1M_2P_2$ are not similar, and equation (1) does not hold. Hence equation (1) holds for all points on the line and for no others. It is therefore the equation of the line.

The equation may be solved for y and written

$$y=(-\tfrac{5}{8})x+\tfrac{7}{8}.$$

The equation is the law of the dependence of y upon x. It may be stated as follows: The ordinate of any point on the straight line passing through $(3, -1)$ and $(-5, 4)$ is equal to $-\frac{5}{8}$ of the abscissa of the point plus $\frac{7}{8}$.

Equation (1) might also be solved for x, which would express x as a function of y.

EXAMPLE 2. Consider the locus of a point which moves so as to keep always at a distance 6 from the point $P_1(3, 2)$.

The locus is a circle with radius 6 and with center at (3, 2).

Here again the value of the ordinate of any point on the locus is a function of the abscissa of the point. To find the law that expresses the ordinate as a function of the abscissa, consider any point $P(x, y)$ on the circle. The condition that P must fulfill is that

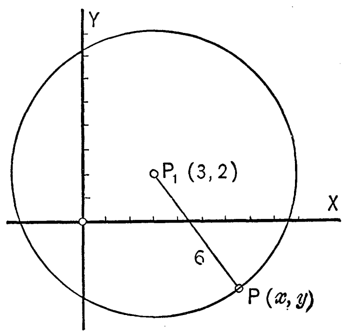


FIG. 41.

$$P_1P=6.$$

Now $$P_1P = \sqrt{(x-3)^2+(y-2)^2}.$$

$$\therefore \quad (x-3)^2+(y-2)^2 = 36. \tag{2}$$

Since eq. (2) is true for all points on the circle and for no others, it is the equation of the locus.

If the equation be solved for y, the result is

$$y = 2 \pm \sqrt{36-(x-3)^2}.$$

This equation expresses y as a function of x.

Since there are two values of y for every value of x, y is a double-valued function of x.

Equation (2) might be solved for x, and x be thus expressed as a function of y.

Example 3. A point moves in the plane so as to keep equidistant from $P_1(3, -2)$ and $P_2(-4, 7)$; to find the equation of the locus.

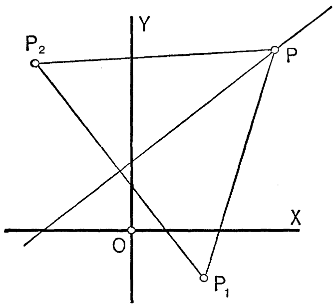


Fig. 42.

To find the equation of the locus, one must express by means of an equation which contains the coördinates of any point of the locus that geometric condition which is satisfied by all points of the locus and by no others. This property is expressed by the equation

$$P_1P = P_2P.$$

Expressed in terms of the coordinates of the point P, this equation becomes

$$\sqrt{(x-3)^2+(y+2)^2} = \sqrt{(x+4)^2+(y-7)^2}. \tag{1}$$

Squaring both members, cancelling, and collecting, there results

$$7x - 9y + 26 = 0, \tag{2}$$

which is the desired equation of the locus. For all values of x and y that satisfy (1) also satisfy (2). In retracing the steps from (2) to (1), a double sign is introduced which would give

$P_1P = \pm P_2P$. But as P_1P and P_2P are positive distances, the equation containing the minus sign has no geometric significance. Equations (1) and (2) therefore are satisfied by precisely the same points.

The locus is known from plane geometry to be the straight line which is perpendicular to P_1P_2 at its middle point.

Example 4. A point moves so that the sum of its distances from $P_1(4, 0)$ and $P_2(-4, 0)$ is always equal to 10; to find the equation of the locus.

Let P (x,y) be any point of the locus. The geometric condition satisfied by all points of the locus and by no others is expressed by the equation

$$P_2P + P_1P = 10.$$

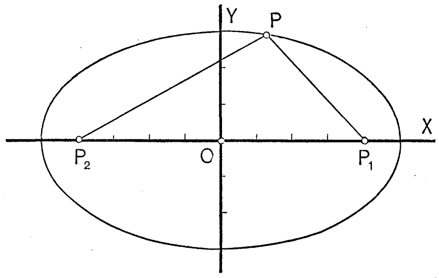


Fig. 43.

Expressed in terms of the coördinates of the point P, this becomes

$$\sqrt{(x-4)^2 + y^2} + \sqrt{(x+4)^2 + y^2} = 10.$$

When freed from radicals, this equation becomes

$$9x^2 + 25y^2 = 225.$$

This is the equation of the locus. It will be shown in Art. 83 that no new points are introduced into the locus by squaring.

A point which moves so that the sum of its distances from two fixed points is constant, describes an **ellipse**.

The above locus is therefore an ellipse.

Points of the locus may be obtained by describing arcs with P_1 and P_2 as centers and radii whose sum is 10. The intersections of two such arcs are points of the locus.

Example 5. A point moves so that the difference of its distances from $P_1(5, 0)$ and $P_2(-5, 0)$ is 8; to find the equation of the locus.

Let $P(x, y)$ be any point of the locus.

The geometric condition satisfied by all points of the locus and by no other points is then

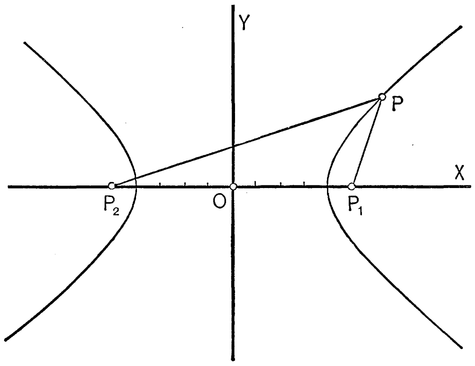


Fig. 44.

$$P_1P - P_2P = \pm 8.$$

This equation when expressed in terms of x and y and freed from radicals reduces to

$$9x^2 - 16y^2 = 144,$$

which is the equation of the given locus. It will be shown in Art. 87 that no new points are introduced into the locus by squaring.

A point which moves so that the difference of its distances from two fixed points is constant, describes an **hyperbola**.

The above locus is therefore an hyperbola.

Points of the locus may be obtained by describing arcs with P_1 and P_2 as centers and radii whose difference is 8. The points of intersection of two such arcs are points of the locus.

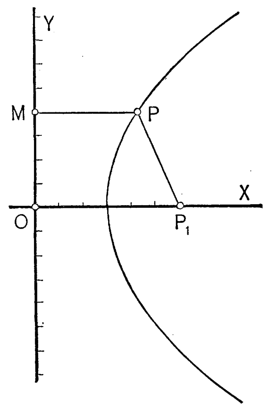


Fig. 45.

Example 6. A point moves so that it remains always equidistant from P_1 (6, 0) and the y-axis; to find the equation of the locus.

Let $P(x, y)$ be any point of the locus. From P draw PM perpendicular to OY. Then the geometric condition to be satisfied by P is

expressed by the equation

$$MP = P_1P.$$

Expressed in terms of the coördinates of $P(x, y)$ this is

$$x = \sqrt{(x-6)^2 + y^2}, \tag{1}$$

which on squaring reduces to

$$y^2 = 12\,x - 36. \tag{2}$$

This is the equation of the locus.

That no new points were introduced into the locus by squaring eq. (1) may be seen as follows: Any values of x and y that satisfy (1) also satisfy (2), but there are values of x and y that satisfy (2) that do not satisfy (1). For in retracing the steps from (2) to (1) a double sign is introduced; *i.e.* given eq. (2), there follows

$$x = \pm\sqrt{(x-6)^2 + y^2}.$$

Now it is evident geometrically that no point can be equidistant from the y-axis and $(6, 0)$ and have its abscissa negative. Therefore only the plus sign can be used. Therefore all points whose coördinates satisfy (2) also satisfy (1). No real values of x and y could therefore have been introduced into eq. (1) by squaring.

A point which moves so as to keep equidistant from a fixed point and a fixed straight line describes a **parabola**.

The above locus is therefore a parabola.

44. Method of finding the equation of the locus of points which satisfy a given condition. In finding the equation of the locus of points satisfying a given condition, a certain method was followed in the preceding examples. This method will suffice for finding the equation of the locus of points satisfying any condition, if that condition can be expressed by means of an equation. The method may be formulated as follows:

To find the equation of the locus of points which satisfy a given condition,

(1) *Assume any point P on the locus.*

(2) *Write the equation which expresses the condition that P must satisfy.*

(3) *Express this equation in terms of the coördinates of P and simplify the equation.*

45. Intercepts of a locus on the axes. The abscissa of a point where a locus cuts the x-axis is called an ***x*-intercept** of the locus. The ordinate of a point where a locus cuts the y-axis is called a ***y*-intercept** of the locus.

If the equation of the locus is known, the x-intercepts may be found by letting y equal zero in the equation and solving the resulting equation for x. Likewise the y-intercepts may be found by letting x equal zero in the equation and solving the resulting equation for y.

EXERCISE XV

Derive the equations of the following loci. Find the intercepts of the loci on the axes. Plot the loci.

1. A straight line through $(1, 4)$ and $(-6, 7)$.
2. A straight line through the origin making an angle of $60°$ with the x-axis.
3. The x-axis. The y-axis. A parallel to the x-axis through $(5, 2)$.
4. A straight line through $(3, -5)$ with slope 2.
5. A straight line through $(a, 0)$ and $(0, b)$.
6. A straight line through $(0, b)$ with slope m.
7. A circle with radius 5 and center at $(2, -4)$.
8. A circle with center at $(-6, 4)$ and passing through $(3, 1)$.
9. A circle with the ends of a diameter at $(5, -6)$ and $(3, 12)$.
10. A circle with center at (h, k) and radius r.
11. A circle with center at the origin and radius r.
12. A circle tangent to both axes and radius r.
13. A circle tangent to the y-axis at the origin and radius r.

14. The locus of a point which moves so that the sum of its distances from (0, 3) and (0, − 3) is 8.

15. The locus of a point which moves so that the difference of its distances from (0, 3) and (0, − 3) is 4.

16. The locus of a point which moves so as to remain always equidistant from the point (0, − 4) and the x-axis.

17. The locus of a point which moves so that the sum of its distances from (3, 2) and (− 6, 1) is 12.

18. The locus of a point which moves so that the difference of its distances from (2, 3) and (− 5, − 1) is 6.

19. The locus of a point which moves so as to keep equally distant from (− 3, 4), and the line parallel to the y-axis through (8, 6).

20. The perpendicular bisector of the line joining (1, 7) and (8, 2).

21. A column of concrete 50 in. long was compressed longitudinally and the following numbers obtained, in which $P =$ number of pounds compression per square inch of cross section of the column, and $\epsilon =$ number of inches of compression, the initial load being 100 lb. per square inch.

P	100	150	200	300	400	500	550	
ϵ	0	.0007	.0015	.0034	.0057	.0080	.0093	
P	600	600	650	700	800	900	1000	column failed.
ϵ	.0108	.0112	.0121	.0139	.0175	.0221	.0275	

Make a graph which shows P as a function of ϵ, and get what information you can from the curve.

22. A steel rod of diameter .564 in., length 3 in., was subjected to a tensile force. The following measurements were made, in which

$P =$ number of pounds tension per square inch of cross section of the rod,
$\lambda =$ number of inches extension, the initial load being 1000 lb. per square inch.

P	1000	5000	10,000	20,000	30,000	40,000	36,000	37,000
λ	0	.0003	.0008	.0018	.0028	.0039	.0058	.0072
P	38,000	39,000	40,009	41,000	42,000	44,000	46,000	50,000
λ	.0114	.0559	.0596	.0615	.0669	.0800	.0905	.1210

Make a graph which shows P as a function of λ. What information do you get from the curve?

23. The following measurements were taken in an experiment in which an india rubber cord was stretched by hanging a weight to its end.

$W =$ weight in kilograms, $L =$ length in centimeters.

W	0	.5	1.0	1.5	2.0	2.5	3.0	3.5
L	10	10.1	10.3	10.6	10.9	11.3	11.7	12.2
W	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
L	12.7	13.3	13.9	14.6	15.3	16.1	16.9	17.9

Make a graph which shows W as a function of L.

24. In Ex. 23 reduce W to pounds and L to inches, and draw the graph. How does the curve compare with that of Ex. 23? By what choice of scale units could you make the two curves coincide?

CHAPTER IV

LOCUS OF AN EQUATION

46. The second fundamental problem. In the preceding chapter some equations of simple loci were obtained from the geometric conditions which the points of the loci satisfied. In this chapter the converse problem of finding the locus when the equation is given will be considered for some simple equations.

ILLUSTRATIONS. EXAMPLE 1. To find the locus of the equation
$$y = 2x + 1.$$

Any number of points whose coördinates satisfy this equation may be found; for any value may be assigned to x and a corresponding value for y computed from the equation. A few corresponding values so obtained are:

$$\begin{array}{lllllll} x & 0, & 1, & 2, & 4, & -3, & -\frac{13}{2}, \\ y & 1, & 3, & 5, & 9, & -5, & -12. \end{array}$$

Plot the points determined by these pairs of values of x and y. They seem to lie on a straight line.

That the locus of the equation is a straight line may be proved as follows:

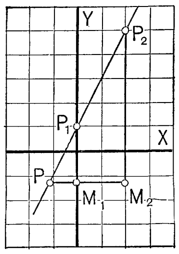


FIG. 46.

Draw a straight line through two points whose coördinates satisfy the equation, as P_1 (0, 1) and $P_2(2, 5)$. (Fig. 46.) Take any point $P(x, y)$ on this line and through it draw a line parallel to the x-axis. From P_1 and P_2 drop perpendiculars to this line, meeting it in M_1 and M_2.

Then from similar triangles, PM_1P_1 and PM_2P_2,

$$\frac{M_2P_2}{M_2P} = \frac{M_1P_1}{M_1P}, \text{ or } \frac{5-y}{x-2} = \frac{1-y}{x-0},$$

which reduces to

$$y = 2x + 1.$$

This equation, therefore, holds for every point on the line.

Conversely, all points whose coördinates satisfy the equation lie on the line; for if a point $P(x, y)$ be taken not on the line, the triangles PM_1P_1 and PM_2P_2 are not similar, and hence the above equation does not hold.

Hence the equation $y = 2x + 1$ is satisfied by all points on the straight line through (0, 1) and (2, 5) and by no others. The line is therefore the locus of the equation.

Example 2. To find the locus of the equation,

$$x^2 + y^2 - 6x + 8y = 24.$$

This equation may be brought into a form like that of eq. (2) of Art. 43, by completing the squares in the terms containing x and in those containing y as follows,

$$x^2 - 6x + 9 + y^2 + 8y + 16 = 24 + 9 + 16,$$

or

$$(x - 3)^2 + (y + 4)^2 = 49.$$

Now the left-hand member of this equation is equal to the square of the distance from (x, y) to $(3, -4)$, and the equation therefore states that this distance is equal to 7. Hence (x, y) must lie on the circumference of a circle with center at $(3, -4)$ and radius 7.

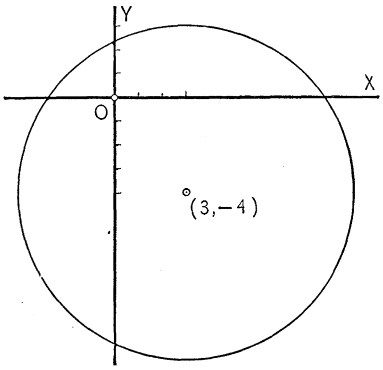


Fig. 47.

Moreover, the coördinates of any point on this circle satisfy the equation. Hence the circle is the locus of the equation. (Fig. 47.)

Example 3. To plot the locus of

$$y^2 = 4x.$$

The following pairs of values of x and y are obtained by arbitrarily assigning values to x and computing the corresponding values of y.

x	0,	$\frac{1}{4}$,	1,	2,	3,	4,	5,	6,	10,
y	0,	± 1,	± 2,	$\pm\sqrt{8}$,	$\pm\sqrt{12}$,	± 4,	$\pm\sqrt{20}$,	$\pm\sqrt{24}$,	$\pm\sqrt{40}$.

From the equation the following facts are readily seen to be true:

(1) If x is negative, y is imaginary; therefore no part of the locus lies to the left of the y-axis.

(2) Every positive value of x gives two values of y which differ only in sign; therefore the points of the locus lie in pairs such that the x-axis bisects at right angles the lines joining the pairs.

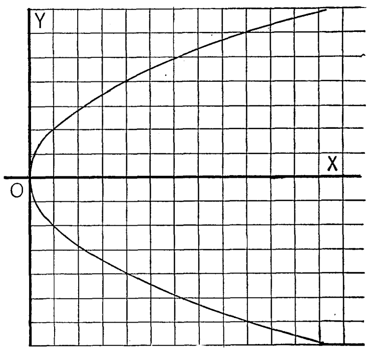


FIG. 48.

(3) As x increases, the positive value of y also increases, and as x becomes infinite, y also becomes infinite; the locus therefore recedes indefinitely from both axes as x increases indefinitely.

(4) A small change in x makes a small change in y.

The part of the locus which lies in the first quadrant may, therefore, be thought of as generated by a moving point which, starting at the origin, moves along a curve gradually rising as the point moves to the right and passing through the above calculated points.

The part of the locus which lies below the x-axis could be obtained from that above the x-axis by folding the upper part of the plane over upon the lower part, using the x-axis as an axis of revolution.

The locus is therefore approximately the curve of Fig. 48.

EXERCISE XVI

Prove that the locus of each of the equations from 1 to 5 is a straight line. Find the intercepts of the lines on the axes and draw the lines.

1. $3x - 4y = 6$.
2. $2x + 5y = 12$.
3. $\frac{x}{3} + \frac{y}{4} = 1$.
4. $y = 7x + 3$.
5. $4x - 8y + 9 = 0$.

Prove that the locus of each of the following equations is a circle, and find the center and radius.

6. $x^2 + y^2 - 4x = 0$.
7. $x^2 + y^2 - 8x + 2y = 8$.
8. $x^2 + y^2 = r^2$.
9. $x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$.
10. $x^2 + y^2 + x - 3y = 1$.
11. $x^2 + y^2 - 2ax = 0$.

Plot the loci of the following equations:

12. $y^2 = 4(x - 2)$.
13. $x^2 = 6y$.
14. $x^2 = 8(y - 4)$.
15. $y^2 = -4x$.
16. $x^2 = -y$.
17. $x - 3 = 2(y + 1)^2$.
18. $y^2 = mx$, letting $m = \frac{1}{10}$, 1, 4, 16, 100, -1, -100.
19. $x^2 = my$, letting m take different values.
20. $x^2 + 4y^2 = 16$.
21. $x^2 - 4y^2 = 16$.

47. Symmetry. Before taking up more difficult problems in loci it will be well to discuss briefly the subject of symmetry of a curve with respect to a line and with respect to a point.

Two points are said to be **symmetric with respect to a** given **line** when the given line bisects at right angles the line joining the two points.

Two points are said to be **symmetric with respect to a** given **point** when the given point bisects the line joining the two points.

A locus of points is said to be **symmetric with respect to a** given **line** when all points of the locus lie in pairs which are symmetric with respect to the given line.

The line is then called an **axis of symmetry**.

A locus of points is said to be **symmetric with respect to a** given **point** when all points of the locus lie in pairs which are symmetric with respect to the given point.

The given point is then called a **center of symmetry**.

Illustrations. (*a*) The points (x, y) and $(-x, y)$ are sym-

metric with respect to the y-axis, the points (x, y) and $(x, -y)$ are symmetric with respect to the x-axis, and the points (x, y) and $(-x, -y)$ are symmetric with respect to the origin.

(*b*) In $y^2 = 4x$, if (x, y) is a point of the locus, so also is $(x, -y)$; for if the coördinates of either point satisfy the equation, so do the coördinates of the other. The locus is therefore symmetric with respect to the x-axis.

(*c*) In $x^2 + 4y^2 = 16$, if (x, y) is a point on the locus, so are $(-x, y)$, $(x, -y)$, and $(-x, -y)$; for if the coördinates of the first point satisfy the equation, so do the coördinates of each of the other points. The locus is therefore symmetric with respect to the y-axis, with respect to the x-axis, and with respect to the origin.

48. Tests for symmetry with respect to the coördinate axes and the origin. If an equation is such that it is unchanged by replacing x by $-x$, the locus of the equation is symmetric with respect to the y-axis. For, whatever value, say x_1, be given to x, the resulting equation which determines the corresponding value, or values, of y will be the same equation as that obtained by substituting $-x_1$ for x. Hence x_1 and $-x_1$ give the same values of y.

Similarly, if replacing y by $-y$ leaves the equation unchanged, the locus is symmetric with respect to the x-axis.

If replacing x by $-x$ and y by $-y$ leaves the equation unchanged, the locus is symmetric with respect to the origin.

In particular, if an equation contains only even powers of x, the locus is symmetric with respect to the y-axis. If it contains only even powers of y, the locus is symmetric with respect to the x-axis. If the terms of an equation are all of even degree, or are all of odd degree in x and y, the locus is symmetric with respect to the origin. (In applying this last test a constant term must be considered as of even degree.)

49. Discussion of an equation. When it is desired to plot the locus of an equation in two variables, it is well to discover

as many properties and facts concerning the locus as one can by a study of the equation. Some important things to look for are

(1) Symmetry.

(2) Points where the locus crosses the axes.

(3) What values, if any, of one variable make the other imaginary.

(4) What finite values, if any, of one variable make the other infinite.

(5) How increasing or decreasing one variable will affect the other.

(6) What value, if any, does one variable approach when the other variable becomes infinite.

50. Illustrations. Example 1. To plot the locus of

$$x^2 + 4y^2 = 16. \qquad (1)$$

If the equation be solved for x and y, respectively, there results

$$x = \pm 2\sqrt{4 - y^2} \qquad (2)$$

and

$$y = \pm \tfrac{1}{2}\sqrt{16 - x^2}. \qquad (3)$$

(1) Equation (1) shows the curve to be symmetric with respect to both coördinate axes and the origin.

(2) If $y = 0$, $x = \pm 4$; if $x = 0$, $y = \pm 2$. Hence the curve meets the axes at $(4, 0)$, $(-4, 0)$, $(0, 2)$, and $(0, -2)$.

(3) Equation (2) shows that if $y^2 > 4$, x is imaginary. $\therefore y$ cannot be greater than 2 nor less than -2.

Likewise, eq. (3) shows that x cannot be greater than 4 nor less than -4.

(4) No finite value of either variable can make the other infinite.

(5) From eq. (3) it is clear that as x increases gradually from 0 to 4, taking all values in that interval, the value of y represented by the positive radical steadily decreases from 2 to 0.

(6) Values of x and y are excluded from becoming infinite by (3).

The part of the locus that lies in the first quadrant may then be thought of as generated by a point which, starting at (0, 2), moves gradually to the right and downward until it reaches (4, 0). A few additional points through which the curve passes will then suffice for a fairly accurate drawing of the curve. A few points computed from eq. (3) are

x	1	2	3	3.5,
y	1.9	1.7	1.3	.96.

The curve is therefore approximately as shown in Fig. 49.

The curve is an ellipse, as will be shown later.

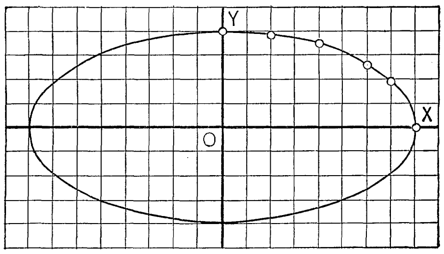


FIG. 49.

EXAMPLE 2. To plot the locus of

$$x^2 - 4y^2 = 16. \qquad (1)$$

Solving for x and y, respectively,

$$x = \pm 2\sqrt{y^2 + 4}, \qquad (2)$$

$$y = \pm \tfrac{1}{2}\sqrt{x^2 - 16}. \qquad (3)$$

(1) Equation (1) shows the curve to be symmetric with respect to both coördinate axes and the origin.

(2) If $x = 0$, y is imaginary; if $y = 0$, $x = \pm 4$. Hence the locus does not meet the y-axis, and meets the x-axis in (4, 0) and $(-4, 0)$.

(3) From eq. (2) it is evident that x is real for all real values

of y, and from eq. (3) that y is imaginary for all values of x between -4 and 4, and is real for all other values of x.

(4) No finite values of either variable makes the other infinite.

(5) Considering the value of y corresponding to the positive sign of the radical in eq. (3), and considering positive values of x, it is evident that as x increases y also increases, a small change in x making a small change in y.

(6) As x increases indefinitely, y also increases indefinitely.

Moreover, as x becomes larger and larger, $\sqrt{x^2-16}$ differs less and less from x. This may be proved as follows:

The difference between x and $\sqrt{x^2-16}$, *i.e.* $x-\sqrt{x^2-16}$, may be expressed as

$$x-\sqrt{x^2-16}=\frac{(x-\sqrt{x^2-16})(x+\sqrt{x^2-16})}{x+\sqrt{x^2-16}}=\frac{16}{x+\sqrt{x^2-16}}.$$

Now, when x increases indefinitely, this fraction decreases indefinitely and approaches the limiting value 0. Therefore as x increases indefinitely, the value of y approaches nearer and nearer without limit to the value of $\frac{1}{2}x$.

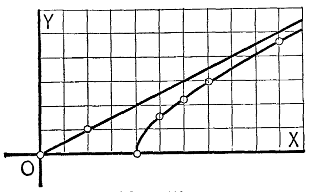


FIG. 50.

Now, $y=\frac{1}{2}x$ is easily shown to be the equation of a straight line through the origin and the point (2, 1). Let this line be drawn. (Fig. 50.) The curve will then come nearer and nearer without limit to this line as x becomes infinite.

A few points through which the curve passes in the first quadrant are

x	4	5	6	7	10,
y	0	1.5	2.2	2.9	4.6.

The part of the locus which lies in the first quadrant may then be thought of as generated by a point which, starting at

(4, 0), gradually rises as it moves to the right, passes through the above points, and approaches nearer and nearer to the straight line whose equation is $y = \frac{1}{2}x$. (Fig. 50.)

The complete locus is obtained from the part in the first quadrant by considerations of symmetry. (Fig. 51.)

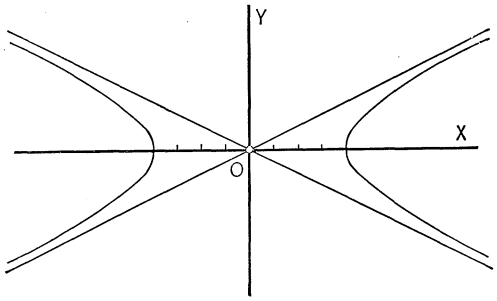


FIG. 51.

The curve is an hyperbola, as will be proved later.

The straight line to which the curve approaches indefinitely near as the point generating the curve recedes indefinitely is called an **asymptote** of the curve.

EXAMPLE 3. To plot the locus of

$$y = \frac{2x+1}{x-3}. \qquad (1)$$

Solving for x,
$$x = \frac{3y+1}{y-2}. \qquad (2)$$

(1) The locus is not symmetric with respect to either coördinate axis or the origin.

(2) If $x = 0$, $y = -\frac{1}{3}$; if $y = 0$, $x = -\frac{1}{2}$. $\therefore$ the curve meets the axes in $(0, -\frac{1}{3})$ and $(-\frac{1}{2}, 0)$.

(3) No real values of either variable make the other imaginary.

(4) If $x = 3$, y is infinite; if $y = 2$, x is infinite.

(5) By division, eq. (1) may be written

$$y = 2 + \frac{7}{x-3}. \qquad (3)$$

From this equation it follows that as x increases from a numerically large negative number to 3, y steadily decreases from

a value a little less than 2 to $-\infty$. As x increases through 3, y changes from $-$ to $+$, and as x increases from 3, y steadily decreases and approaches the limiting value 2 when x becomes infinite.

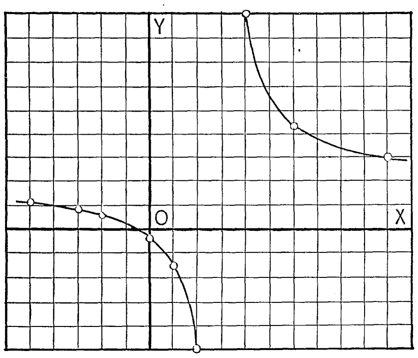


Fig. 52.

The following points are on the locus:

x	-5	-3	-2	0	1	2	4	6	10,
y	$\frac{9}{8}$	$\frac{5}{6}$	$\frac{3}{5}$	$-\frac{1}{3}$	$-\frac{3}{2}$	-5	9	$\frac{13}{3}$	3.

The curve may then be sketched as in Fig. 52. The lines $x=3$ and $y=2$ are asymptotes of the curve.

Example 4. To plot the locus of

$$y=x(x+1)(x+2).$$

(1) The locus is not symmetric with respect to either coordinate axis or the origin.

(2) The locus meets the axes in $(0, 0)$, $(-1, 0)$, and $(-2, 0)$.

(3) No real values of either variable make the other imaginary.

(4) No finite value of either variable makes the other infinite.

(5) Let x take a numerically large negative value; then y is numerically large, but negative. As x increases from the value

assigned toward -2, each of the factors of y remains negative, but decreases in numerical value; y therefore remains negative, but decreases in numerical value until $x=-2$, when $y=0$. As x passes through the value -2, the factor $x+2$ changes sign and becomes positive, the other factors of y remaining negative in sign until $x=-1$; therefore y is positive for all values of x between -2 and -1. As x passes through -1, y passes through 0 and remains negative for all values of x between -1 and 0. As x increases through 0, y again becomes positive and steadily increases as x increases and becomes infinite when x becomes infinite.

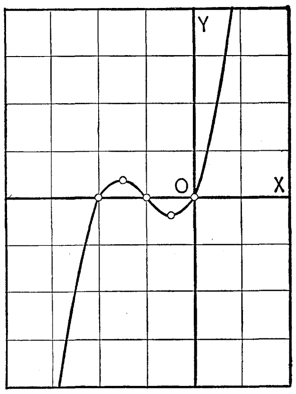


FIG. 53.

The locus may then be generated by a point which, starting indefinitely far to the left and below the origin, steadily rises as it moves to the right until, after crossing the x-axis at $(-2, 0)$, it turns at some value of x between -2 and -1, descends to cross the x-axis at $(-1, 0)$, turns again at some value of x between -1 and 0 and ascends to cross the x-axis at $(0, 0)$, and continually thereafter moves to the right and upward, receding indefinitely from both axes.

The following points are on the curve:

x	-8	-6	-4	-3	-2	$-\frac{3}{2}$,
y	-336	-120	-24	-6	0	$\frac{3}{8}$.
x	-1	$-\frac{1}{2}$	0	1	2	6,
y	0	$-\frac{3}{8}$	0	6	24	336.

The curve is shown in Fig. 53.

EXAMPLE 5. To plot the locus of

$$y^2=(x+2)(x-1)(x-3).$$

(1) The locus is symmetric with respect to the x-axis.

(2) The locus crosses the x-axis at $(-2, 0)$, $(1, 0)$, $(3, 0)$, and the y-axis at $(0, +\sqrt{6})$ and $(0, -\sqrt{6})$.

(3) If x is less than -2, or is between 1 and 3, y is imaginary.

(4) No finite value of either variable makes the other infinite.

(5) Since $y^2 = 0$ when $x = -2$ and when $x = 1$, and is positive for all values of x between -2 and 1, therefore as x increases from -2 to 1, the positive value of y must increase from 0 when $x = -2$ and then decrease to 0 when $x = 1$.*

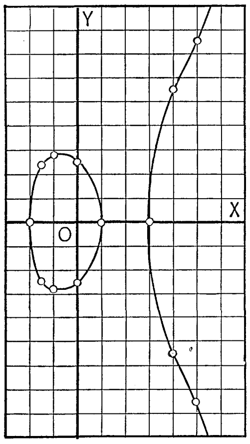


FIG. 54.

As x increases from 1 to 3, y^2 is negative; y is imaginary.

As x increases from 3, y^2 becomes and remains positive and steadily increases as x increases. The positive value of y, therefore, increases as x increases from 3.

(6) When x becomes infinite, y becomes infinite.

The curve then consists of a closed portion between $x = -2$ and $x = 1$, and an infinite branch to the right of $x = 3$.

The following points are on the curve:

x	-2	$-\frac{3}{2}$	-1	0	1	3	4	5	6	7	10,
y	0	± 2.4	± 2.8	± 2.5	0	0	± 5.5	± 7.5	± 11	± 14.7	± 27.

The curve is shown in Fig. 54.

* At present the student has no means of telling that y does not change from increasing to decreasing and from decreasing to increasing several times as x increases from -2 to 1; nor of telling where a change of this kind takes place. The investigation of such questions will be the subject of a later chapter.

EXERCISE XVII

Discuss the following equations and plot the curves:

1. $xy = 4$.
2. $y = 4x^2$.
3. $y^2 = -x$.
4. $y = -x^2$.
5. $y^2 = x^2$.
6. $y = x(x-3)$.
7. $y = \dfrac{x}{x-3}$.
8. $y = (x+5)x(x-3)$.
9. $y^2 + 4x^2 = 4$.
10. $y^2 - 4x^2 = 4$.
11. $y^2 = 4 - x^2$.
12. $y^2 = x^3$.
13. $y = x^3$.
14. $(x+2)(y+3) = 1$.
15. $y = 4x^2 + 4$.
16. $\dfrac{1}{y} = \dfrac{x+1}{x-2}$.
17. $x^2y^2 = 4$.
18. $y^2 = (x-1)(x-3)(x-6)$.
19. $y^2 = (x-1)^2(x-2)$.
20. $y = \dfrac{1}{(x-1)(x-4)}$.
21. $y = \dfrac{x+2}{(x-1)(x-3)}$.
22. $y = \dfrac{(x+2)(x-5)}{(x+1)(x-3)}$.
23. $y = \dfrac{2(x+3)(x-2)}{(x+1)(x-4)}$.
24. $pv =$ a constant.
25. $pv^{1.2} = 6$.
26. $v = 32t$.
27. $s = 16t^2$.
28. $y - 2 = \dfrac{1}{x-3}$.

29. A light is placed at a distance h ft. above a plane surface. Given that the illumination of the plane at any point varies inversely as the square of the distance from the light, and directly as the cosine of the angle between the incident rays and the perpendicular to the plane; prove that the illumination at a point in the plane at a distance x from the foot of the perpendicular from the light to the plane is given by

$$\mathrm{I} = \frac{Ch}{(x^2 + h^2)^{\frac{3}{2}}}, \text{ where } C \text{ is a constant.}$$

Plot the curves for $h = 20$ ft., 30 ft., and 40 ft.

CHAPTER V

TRANSFORMATION OF COÖRDINATES

51. Change of axes. The cöordinates of a point in the plane depend upon the position of the axes to which the coördinates are referred.

A change of axes will change the coördinates. The equations connecting the coördinates of any point in the plane with the coördinates of the same point when referred to another system will next be derived for certain changes of axes.

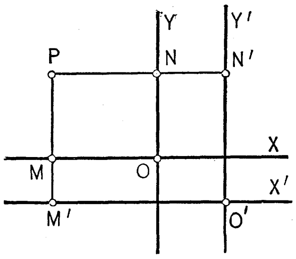


FIG. 55.

52. Translation of axes. Assume a set of axes OX and OY and a second set $O'X'$ and $O'Y'$ parallel respectively to the first axes. Let O' referred to OX and OY be (h, k).

Take any point P in the plane and let its coördinates referred to OX and OY be x and y, and referred to $O'X'$ and $O'Y'$ be x' and y'. Then (see Fig. 55),

$$x = NP, \quad x' = N'P, \quad h = NN',$$
$$y = MP, \quad y' = M'P, \quad k = MM'.$$

Now $$NP = NN' + N'P,$$
and $$MP = MM' + M'P,$$
or $$\boldsymbol{x = h + x'}.$$
$$\boldsymbol{y = k + y'}.$$

This transformation from one set of axes to the other is called "Translation of the axes."

53. Rotation of axes. Let the rectangular axes OX' and OY' make an angle θ with OX and OY respectively. Let any point P have coördinates (x, y) referred to OX and OY, and (x', y') referred to OX' and OY'. Let $OP = r$ and $\angle X'OP = \phi'$. Then $\angle XOP = \phi' + \theta$.

Then, Fig. 56,

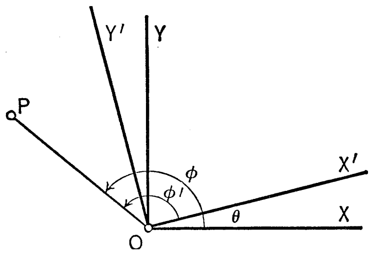


FIG. 56.

$$x' = r \cos \phi',$$
$$y' = r \sin \phi',$$
$$x = r \cos (\phi' + \theta),$$
$$y = r \sin (\phi' + \theta).$$

Expanding the last two equations,

$$x = r \cos \phi' \cos \theta - r \sin \phi' \sin \theta,$$
$$y = r \cos \phi' \sin \theta + r \sin \phi' \cos \theta,$$

or

$$\boldsymbol{x = x' \cos \theta - y' \sin \theta,}$$
$$\boldsymbol{y = x' \sin \theta + y' \cos \theta.}$$

These equations hold for any point in the plane. They express x and y in terms of x' and y'.

To express x' and y' in terms of x and y, these equations may be solved for x' and y', or the equations may be derived as follows:

In Fig. 56, let $\angle XOP = \phi$,

then

$$x = r \cos \phi,$$
$$y = r \sin \phi,$$

$$x' = r\cos(\phi - \theta) = r\cos\phi\cos\theta + r\sin\phi\sin\theta,$$
$$y' = r\sin(\phi - \theta) = r\sin\phi\cos\theta - r\cos\phi\sin\theta,$$

or
$$x' = x\cos\theta + y\sin\theta,$$
$$y' = y\cos\theta - x\sin\theta.$$

This transformation from one set of axes to the other is known as "Rotation of the axes."

54. Applications. The formulas of translation and rotation of axes may be used to simplify equations, thereby making the construction and classification of the loci easier.

EXAMPLE 1. Consider the equation

$$12x^2 - 48x + 3y^2 + 6y = 13.$$

Let the axes be translated to a new origin (h, k), the formulas for which are

$$x = x' + h, \quad y = y' + k.$$

Substituting these values in the equation, it becomes

$$12x'^2 + 3y'^2 + (24h - 48)x' + (6k + 6)y' + 12h^2 + 3k^2 - 48h + 6k - 13 = 0.$$

The quantities h and k may have any real values assigned to them. If they be so chosen that the terms of first degree in x' and y' drop out of the equation, the equation will be simplified and the locus will be symmetric with respect to the axes $O'X'$ and $O'Y'$. To accomplish this it is only necessary to let

$$24h - 48 = 0,$$
$$6k + 6 = 0,$$

from which
$$h = 2, \ k = -1.$$

The equation then becomes

$$12x'^2 + 3y'^2 = 64.$$

This, then, is the equation of the locus referred to the axes $O'X'$ and $O'Y'$.

The equation is now easily discussed and the locus plotted. Fig. 57 shows the locus and both sets of axes.

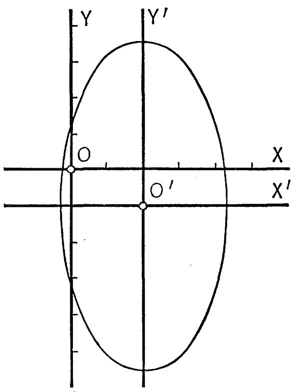


FIG. 57.

The student should discuss and plot the locus.

EXAMPLE 2.

$$y^2 - 8y + 4x + 6 = 0.$$

Translate the axes to the new origin (h, k) by means of

$$x = x' + h,\ y = y' + k.$$

The transformed equation is

$$y'^2 + 2ky' + k^2 - 8y' - 8k + 4x' + 4h + 6 = 0.$$

Here it is not possible to choose h and k so that the terms of first degree in x' and y' will drop out, since the coefficient of x' is 4. They can, however, be so chosen that the term in y' and the constant term will drop out. To accomplish this it is only necessary to let

$$2k - 8 = 0,$$

and $k^2 - 8k + 4h + 6 = 0$, from which $k = 4,\ h = \frac{5}{2}$.

The equation then becomes

$$y'^2 + 4x' = 0.$$

The locus is now easily constructed. (Fig. 58.)

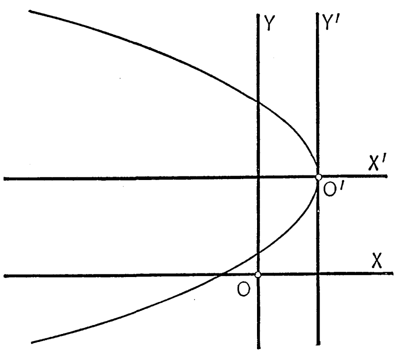


FIG. 58.

EXAMPLE 3.

$$11x^2 + 24xy + 4y^2 = 20. \quad (1)$$

In equations of this form, *i.e.* equations of second degree in x and y containing a term in the product xy, the term in xy may be made to drop out by a rotation of axes.

Let
$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta.$$

Substituting in eq. (1),

$$11(x'\cos\theta - y'\sin\theta)^2 + 24(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$
$$+ 4(x'\sin\theta + y'\cos\theta)^2 = 20. \qquad (2)$$

Expanding and collecting,

$$\begin{array}{r|l|l|l}
11\cos^2\theta & x'^2 - 22\sin\theta\cos\theta & x'y' + 11\sin^2\theta & y'^2 = 20. \quad (3)\\
+24\cos\theta\sin\theta & \quad +24\cos^2\theta & \quad -24\sin\theta\cos\theta & \\
+4\sin^2\theta & \quad -24\sin^2\theta & \quad +4\cos^2\theta & \\
 & \quad +8\sin\theta\cos\theta & &
\end{array}$$

It is now possible to choose θ so that the coefficient of $x'y'$ will become zero; for it is only necessary to have

$$24(\cos^2\theta - \sin^2\theta) = 14\sin\theta\cos\theta,$$

or
$$24\cos 2\theta = 7\sin 2\theta,$$

or
$$\tan 2\theta = \tfrac{24}{7}. \qquad (4)$$

To satisfy eq. (4), let 2θ be the angle in the first quadrant whose tangent is $\frac{24}{7}$. Draw the right triangle with sides 24 and 7 as in Fig. 59. The hypotenuse is then 25.

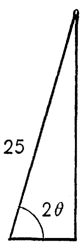


Fig. 59.

$\therefore \quad \sin 2\theta = \frac{24}{25}, \ \cos 2\theta = \frac{7}{25}.$

Now $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, and

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta.$$

$\therefore \quad \sin^2\theta = \frac{9}{25}, \ \cos^2\theta = \frac{16}{25}, \ \sin\theta\cos\theta = \frac{12}{25}.$

Substituting these values in eq. (3) and dividing the resulting equation through by 125, there results

$$4x'^2 - y'^2 = 4.$$

Referred to the new axes the locus is much more easily constructed. The discussion of the equation is very similar to that of example 2, Art. 50.

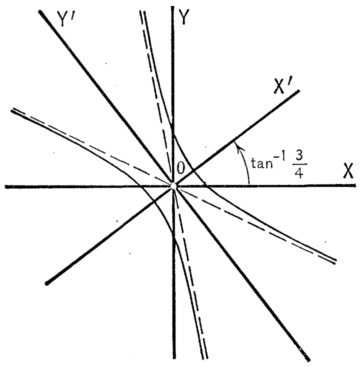


FIG. 60.

The locus and both sets of axes are shown in Fig. 60.

The angle through which the axes are turned is $\tan^{-1}\frac{3}{4}$.

EXERCISE XVIII

Simplify the following equations by a translation of axes to remove the terms of first degree where possible, and by a rotation of axes to remove the terms in xy. Plot the curves and all coördinate axes.

1. $x^2 - 6x + 4y^2 + 8y = 5$.
2. $4x^2 - 4y^2 + x - 2y = 0$.
3. $4x^2 + y^2 - 12x + 2y - 2 = 0$.
4. $x^2 + y^2 - 4x + 2y - 11 = 0$.
5. $y^2 - 6y + 8 = 4x$.
6. $2x^2 - 6y^2 + xy - 5x + 11y = 3$.
7. $x^2 + 2xy + y^2 - 12x + 2y = 3$.
8. $x^2 - xy - 2y^2 - x - 4y - 2 = 0$.
9. $3x^2 + 2xy + 3y^2 = 8$.
10. $xy = 4$.

CHAPTER VI

THE STRAIGHT LINE

55. Theorem. *Every straight line has an equation of first degree in Cartesian coördinates.*

Two cases are to be considered:

(1) The line parallel to a coördinate axis. If the line is parallel to the x-axis, then all points of the line have equal ordinates. $\therefore$ $y = c$, where c is a constant, is true for all points of the line and for no others. It is therefore the equation of the line.

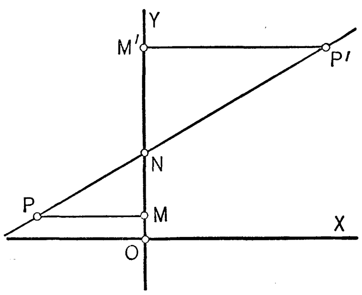


FIG. 61.

Likewise, a line parallel to the y-axis has an equation of the form $x = c$.

(2) The line not parallel to an axis.

Let the line cut the y-axis at $N(0, b)$.

Let $P(x, y)$ be any point of the line. Through P draw PM parallel to the x-axis to meet the y-axis in M. Then as P moves along the line, the ratio $\frac{MN}{PM}$ will remain unchanged. For if P' is any other point of the line, then, by similar triangles,

$$\frac{MN}{PM} = \frac{M'N}{P'M'}.$$

Let this constant ratio be denoted by m.

Then $$\frac{MN}{PM} = m$$

is true for all points on the line, and for no others.

From the figure the values of MN and PM are seen to be $b-y$ and $0-x$ respectively. Therefore

$$\frac{b-y}{0-x}=m,$$

or

$$\boldsymbol{y=mx+b.}$$

This is therefore the equation of the line. It is of first degree in x and y.

For a line passing through the origin, the value of b is zero, and the equation becomes

$$y=mx.$$

The relation between the lines $y=mx$ and $y=mx+b$ is shown in Fig. 62.

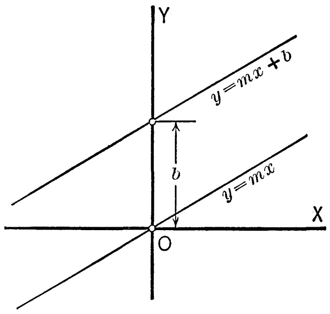


FIG. 62.

It is important to notice that if the axes are rectangular, the constant ratio m, or $\frac{MN}{PM}$, is the slope of the line.

56. The equation of first degree. Conversely, *every equation of first degree in Cartesian coördinates, with real coefficients, is the equation of a straight line.*

The general equation of first degree is of the form

$$Ax+By+C=0. \tag{1}$$

Here again two cases are to be considered:

(1) When either A or B is zero. Suppose $A=0$. Then $B\neq 0$, and the equation may be written

$$y=-\frac{C}{B}.$$

This equation is evidently satisfied by all points on a line parallel to the x-axis, and by no others.

Likewise, if $B = 0$, the equation represents a straight line parallel to the y-axis.

(2) When neither A nor B is zero. Solve the equation for y,

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

Now this is of the same form as

$$y = mx + b,$$

which was found in the preceding article to be the equation of a straight line, and since a straight line can be drawn so that m and b will have any assigned real values, a line can be drawn so that $b = -\frac{C}{B}$, and $m = -\frac{A}{B}$. Then $y = -\frac{A}{B}x - \frac{C}{B}$, or $Ax + By + C = 0$, is the equation of this line.

Hence $$Ax + By + C = 0$$

is the equation of a straight line.

The proofs given in this and the preceding article hold for oblique as well as for rectangular coördinates. It is only in rectangular coördinates, however, that m is the slope of the line.

57. The conditions which determine a straight line. The position of a straight line is determined when there are known either,

(1) Two points on the line,

(2) A point on the line and the direction of the line,

(3) The length and direction of a perpendicular from a fixed point to the line.

Considerations of these conditions lead to the following special forms of the equation of the straight line.

58. The two-point equation. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points. To find the equation of the straight line through them.

Let $P(x, y)$ be any point on the line. Through P_1 draw a line parallel to the y-axis to meet lines parallel to the x-axis through P and P_2 in M and M_2 respectively. Then by similar triangles,

$$\frac{MP_1}{M_2P_1} = \frac{MP}{M_2P_2},$$

which is the same as

$$\frac{\boldsymbol{y - y_1}}{\boldsymbol{y_2 - y_1}} = \frac{\boldsymbol{x - x_1}}{\boldsymbol{x_2 - x_1}}.$$

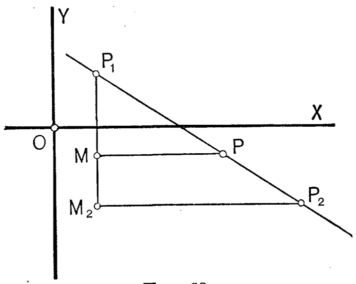


FIG. 63.

This equation holds for every point on the line, and for no others. It is, therefore, the equation of the line.

59. The intercept equation. In the last article let the two given points be $(a, 0)$ and $(0, b)$. The equation then becomes

$$\frac{y - 0}{y - b} = \frac{x - a}{x - 0}.$$

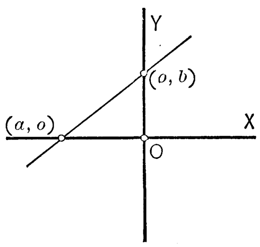


FIG. 64.

Clearing of fractions, transposing, and dividing by ab, the equation reduces to $\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1}$.

This is known as the intercept equation of the line, since a and b are the intercepts of the line on the coördinate axes.

In this equation neither a nor b can be zero.

60. The point-slope equation. Let the line pass through $P_1(x_1, y_1)$ and have a slope m. Let $P(x, y)$ be any point on the line. The slope of the line joining P_1 and P is $\frac{y - y_1}{x - x_1}$, which by hypothesis is equal to m.

$$\therefore \qquad \frac{y - y_1}{x - x_1} = m$$

is an equation which holds for all points on the line and for no others. It is, therefore, the equation of the line.

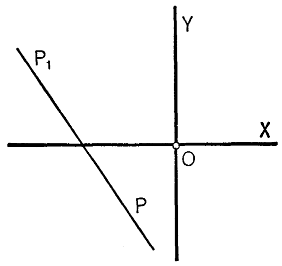


Fig. 65.

Clearing of fractions, it may be written

$$\boldsymbol{y - y_1 = m(x - x_1)}.$$

This equation does not apply to a straight line parallel to the y-axis, for which m is infinite.

61. The slope equation. If in the last article the given point is $(0, b)$, the equation reduces to

$$\boldsymbol{y = mx + b},$$

which is the slope equation already considered. (Art. 55.)

62. The normal equation. Let the distance from the origin to the straight line be p, and let the angle which this perpendicular makes with the x-axis be α. The quantity p will be considered positive always.

Let H be the foot of the perpendicular from the origin to the line. The coördinates of H are then $p \cos \alpha$ and $p \sin \alpha$. The slope of OH is $\tan \alpha$. Therefore the slope of the given line is $-\cot \alpha$.

Hence the line passes through $(p \cos \alpha,\ p \sin \alpha)$ and has a slope equal to $-\cot \alpha$.

The equation of the line is therefore, by Art. 60,

$$y - p \sin \alpha = -\cot \alpha (x - p \cos \alpha).$$

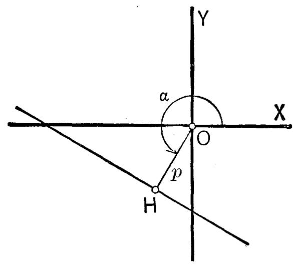


FIG. 66.

Replace $\cot \alpha$ by $\frac{\cos \alpha}{\sin \alpha}$, clear of fractions, and transpose; the equation then becomes

$$x \cos \alpha + y \sin \alpha - p (\cos^2 \alpha + \sin^2 \alpha) = 0,$$

or
$$\boldsymbol{x \cos \alpha + y \sin \alpha - p = 0}.$$

63. Reduction of $Ax + By + C = 0$ to the slope intercept, and normal forms. The equation $Ax + By + C = 0$ may be reduced to the slope, intercept, and normal forms as follows:

(*a*) To reduce $Ax + By + C = 0$ to the slope form. Solve the equation for y. There results

$$y = -\frac{A}{B}x - \frac{C}{B},$$

which is in the form $y = mx + b$, where $m = -\frac{A}{B}$, $b = -\frac{C}{B}$.

The method fails when $B = 0$. The equation cannot then be put in the slope form.

(*b*) To reduce $Ax + By + C = 0$ to the intercept form. Transpose C to the right-hand side of the equation, and divide

by $-C$; the resulting equation may be written

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1,$$

which is in the form $\frac{x}{a} + \frac{y}{b} = 1$, where $a = -\frac{C}{A}$, $b = -\frac{C}{B}$.

The method fails if either A, B, or C is zero. If $C = 0$, both intercepts are zero. If either A or B is zero, the line is parallel to an axis of coördinates.

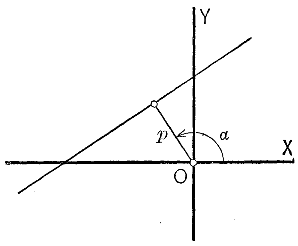


FIG. 67.

(c) To reduce $Ax + By + C = 0$ to the normal form.

Let $x \cos \alpha + y \sin \alpha - p = 0$ be the normal equation of the line. The foot of the perpendicular from the origin to the line is then $(p \cos \alpha,\ p \sin \alpha)$. The coördinates of this point must satisfy the equation $Ax + By + C = 0$.

$$\therefore \qquad Ap \cos \alpha + Bp \sin \alpha + C = 0. \qquad (1)$$

Also the slope of the perpendicular is the negative reciprocal of the slope of the line;

$$\therefore \qquad \tan \alpha = \frac{B}{A}. \quad \text{[See (a) of this article.]} \qquad (2)$$

$$\therefore \qquad \cos \alpha = \frac{1}{\pm\sqrt{1 + \tan^2 \alpha}} = \frac{A}{\pm\sqrt{A^2 + B^2}}.$$

Substituting in (2),

$$\sin \alpha = \frac{B}{\pm\sqrt{A^2 + B^2}}.$$

Substituting these values of $\sin \alpha$ and $\cos \alpha$ in (1),

$$-p = \frac{C}{\pm\sqrt{A^2 + B^2}}.$$

Substituting these values of $\sin\alpha$, $\cos\alpha$, and p in the normal equation of the line, there results

$$\frac{A}{\pm\sqrt{A^2+B^2}}x+\frac{B}{\pm\sqrt{A^2+B^2}}y+\frac{C}{\pm\sqrt{A^2+B^2}}=0.$$

Hence, *the equation of a line is reduced to the normal form by dividing the equation of the line through by the square root of the sum of the squares of the coefficients of x and y. The sign of the radical should be taken opposite to the sign of C so that p will be positive.*

64. Illustration. To reduce $2x-4y+7=0$ to the slope, intercept, and normal forms.

(*a*) Solving for y brings the equation into the slope form

$$y=\tfrac{1}{2}x+\tfrac{7}{4},$$

in which $$m=\tfrac{1}{2},\ b=\tfrac{7}{4}.$$

(*b*) Transposing the constant term, 7, and dividing by -7, brings the equation into the intercept form

$$\frac{x}{-\frac{7}{2}}+\frac{y}{\frac{7}{4}}=1,$$

in which $$a=-\tfrac{7}{2},\ b=\tfrac{7}{4}.$$

(*c*) Dividing through by $-\sqrt{2^2+4^2}$ brings the equation into the normal form

$$-\frac{1}{\sqrt{5}}x+\frac{2}{\sqrt{5}}y-\frac{7}{2\sqrt{5}}=0,$$

in which $$\cos\alpha=-\frac{1}{\sqrt{5}},\ \sin\alpha=\frac{2}{\sqrt{5}},\ p=\frac{7}{2\sqrt{5}}.$$

65. Applications of the formulas. By the use of the formulas derived in this chapter the equations of straight lines which satisfy certain conditions can be easily found.

ILLUSTRATIONS.

(*a*) To find the equation of a straight line which passes through $(3, -5)$ and makes an angle of 30° with the x-axis.

The slope of the required line is tan 30°, or $\frac{1}{\sqrt{3}}$. By substituting in the equation $y - y_1 = m(x - x_1)$ there results

$$y + 5 = \frac{\sqrt{3}}{3}(x - 3),$$

which reduces to

$$\sqrt{3}\,x - 3\,y = 15 + 3\sqrt{3},$$

the required equation.

(b) To find the equation of the straight line which passes through $(-3,\ 1)$ and makes an angle of 60° with the line $4\,x - 9\,y = 12$.

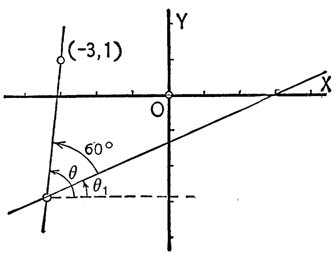


FIG. 68.

Let the angles which the given line and the required line make with the x-axis be θ_1 and θ respectively, and the slopes of these lines be m_1 and m respectively. Then $m_1 = \tan\theta_1$, $m = \tan\ \theta$. But $m_1 = \frac{4}{9}$, and $\theta = \theta_1 + 60°$. (Fig. 68.)

$$\therefore \quad m = \tan\theta = \tan(\theta_1 + 60°)$$

$$= \frac{\tan\theta_1 + \tan 60°}{1 - \tan\theta_1 \tan 60°}$$

$$= \frac{\frac{4}{9} + \sqrt{3}}{1 - \frac{4\sqrt{3}}{9}} = \frac{4 + 9\sqrt{3}}{9 - 4\sqrt{3}}$$

$$= \frac{144 + 97\sqrt{3}}{33}.$$

Therefore the required equation is

$$y - 1 = \frac{144 + 97\sqrt{3}}{33}(x + 3),$$

or approximately

$$y - 1 = 9.45(x + 3).$$

66. The point of intersection of two straight lines. Let the two lines whose equations are

$$A_1x + B_1y + C_1 = 0, \tag{1}$$

and

$$A_2x + B_2y + C_2 = 0, \tag{2}$$

be denoted by L_1 and L_2 respectively. In eq. (1) x and y may be the coördinates of any point on L_1, and in eq. (2) x and y may be the coördinates of any point on L_2, and hence x and y in one equation are not the same in general as x and y in the other. If, however, the lines intersect, there is one pair of values of x and y that satisfy both equations; namely, the coördinates of the point of intersection. Conversely, if values of x and y can be found which satisfy both equations, they are the coördinates of a point on both lines, *i.e.* the point of intersection. Therefore, to find the coördinates of the point of intersection of two lines, solve the equations of the lines as simultaneous.

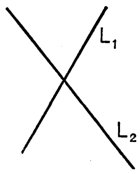


FIG. 69.

What if the lines are parallel?

ILLUSTRATION. To find the point of intersection of

$$3x + 2y = 11$$

and

$$4x - 5y = 7.$$

Solving the equations as simultaneous, the values of x and y are found to be $x = 3$, $y = 1$. Therefore the point of intersection is (3, 1).

Let the student plot the lines and check graphically.

EXERCISE XIX

By substituting in the formulas write the equations of the straight lines which satisfy the following given conditions:

1. Passing through (2, 1) with slope -2.
2. Passing through $(-3, 7)$ and $(2, -5)$.
3. With x- and y-intercepts 3 and -8 respectively.

4. With y-intercept 6 and slope 2.

5. Passing through the origin with slope $-\frac{5}{2}$.

6. With $\alpha = 30^\circ$ and $p = 4$.

7. With $p = 5$ and $m = -\frac{3}{2}$.

8. Passing through $(2, -5)$ parallel to $3x - y + 4 = 0$.

9. Passing through $(0, 0)$ perpendicular to $ax + by + c = 0$.

10. Passing through (x_1, y_1) parallel to $y = mx + b$.

11. With y-intercept b and perpendicular to $Ax + By + C = 0$.

12. Passing through (h, k) parallel to $x \cos\beta + y \sin\beta = q$.

13. Passing through (e, f) parallel to $lx + my + n = 0$.

14. Passing through the origin and perpendicular to $gx + fy = c$.

Reduce, where possible, each of the following equations of straight lines to the intercept, slope, and normal forms, giving the values of a, b, m, α, and p.

15. $3x - 4y = 5$.
16. $y + 2x = 4$.
17. $2x - 5y = 0$.
18. $-x + 2y = 9$.
19. $y - 25 = 0$.

20. Obtain the equation of the straight line which passes through $(1, 2)$ and makes an angle of 60° with the line $2x - 5y = 8$.

21. Two lines, L_1 and L_2, intersect in $(-3, -2)$; L_1 has a y-intercept equal to -6 and makes $\tan^{-1}\frac{3}{2}$ with L_2; find the equations of the two lines.

22. Find the equation of the straight line of slope $-\frac{3}{4}$ which passes through the intersection of $2y - x = 5$ and $x - 3y = 1$.

23. The vertices of a triangle are $(1, 2)$, $(4, -6)$, and $(-5, -3)$; find the equations of its sides.

24. Find the equations of the perpendiculars from the vertices upon the opposite sides of the triangle in example 23, and prove that they meet in a common point.

25. Find the equations of the medians of the triangle in example 23, and prove that they meet in a common point.

26. Find the equation of the line through (h, k) making $\tan^{-1}m$ with $y = lx + b$.

27. A line passes through $(2, 5)$ and is distant 3 from the origin; find its equation. How many solutions?

28. Show that $$Ax + By + C = 0$$
and $$Ax + By + K = 0$$
are parallel, and that
$$Ax + By + C = 0,$$
and $$Bx - Ay + K = 0$$
are perpendicular.

29. What set of lines is obtained by varying b in the equation $y = mx + b$? What set of lines by varying m?

30. Discuss the effect upon the line $Ax + By + C = 0$ of changing each of the constants, keeping the other two unchanged.

31. Find the equation of a straight line which passes through the intersection of $2x - y + 5 = 0$ and $x - 2y + 1 = 0$, and makes an angle of 45° with $y = 2x$.

32. Prove that $ax + by + c + k(lx + my + n) = 0$ is the equation of a straight line which passes through the intersection of $ax + by + c = 0$ and $lx + my + n = 0$. What is the effect on the line of varying k?

33. Using the fact expressed in example 32, find the equation of the straight line which passes through $(3, -1)$ and the intersection of $2x + 4y - 7 = 0$ and $7x - 2y + 13 = 0$, by determining the proper value of k.

34. Find the equation of the straight line which passes through the intersection of $x + 3y - 7 = 0$ and $y - 3x = 2$, and makes an angle of 135° with the x-axis.

35. Find the equation of the straight line which passes through the intersection of $2x - 9y = 18$ and $7y + 5x = 21$, and is parallel to $4x + 6y - 3 = 0$.

36. Find the equation of the straight line perpendicular to $3y = 7x$ which passes through the intersection of $x + 2y = 8$ and $4x = 13y$.

67. Change of sign of $Ax + By + C$. *The expression $Ax + By + C$ is positive for all points on one side of the line $Ax + By + C = 0$, and is negative for all points on the other side of the line.*

Proof. I. Suppose $B \neq 0$. The line is then not parallel to the y-axis. Let L be the line whose equation is $Ax + By + C = 0$.

For all points on this line

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

Let (x_1, y_1) be any point above the line. Since y_1 is greater than the ordinate of the point on the line with abscissa x_1, it follows that

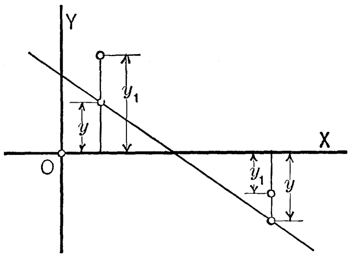


FIG. 70.

$$y_1 > -\frac{A}{B}x_1 - \frac{C}{B},$$

or $$y_1 + \frac{A}{B}x_1 + \frac{C}{B} > 0.$$

This is true, then, for any point above the line. Therefore, for all points above the line,

$$Ax_1 + By_1 + C > 0, \text{ if } B > 0,$$

and $$Ax_1 + By_1 + C < 0, \text{ if } B < 0.$$

In either case the expression has the same sign for all points above the line.

If the point is taken below the line, the inequalities are all reversed. Hence the expression has the same sign for all points below the line, but that sign is opposite to the sign of the expression for points above the line.

II. Suppose $B = 0$. Then $A \neq 0$. The expression becomes $Ax + C$, and the equation of the line becomes $Ax + C = 0$. The line is parallel to the y-axis. On this line $x = -\frac{C}{A}$. To the left of the line $x < -\frac{C}{A}$, and to the right of the line $x > -\frac{C}{A}$. Therefore $Ax + C$ has the same sign for all points on one side of the line and has the opposite sign for all points on the opposite side of the line. Hence the theorem is true in all cases.

An important special case of this theorem is the following: *The sign of the expression* $Ax + By + C$ *is the same as, or opposite to, the sign of* C *according as the point* (x, y) *and the origin are on the same, or opposite, sides of the line* $Ax + By + C = 0$. This follows at once from the theorem, since the value of the expression $Ax + By + C$ is C when the coördinates of the origin are substituted. If $C = 0$ and $A \neq 0$, the student can easily show that the sign of $Ax + By$ is the same as, or opposite to, the sign of A, according as (x, y) lies to the right or left of the line $Ax + By = 0$.

68. Illustration. The expression $3x + 7y - 8$ has the value 2 at (1, 1), which is opposite in sign to C, or -8. Hence (1, 1) and the origin are on opposite sides of the line

$$3x + 7y - 8 = 0.$$

Also the expression $3x + 7y - 8$ has the value -1 at $(2, \frac{1}{7})$, which is the same in sign as -8. Hence $(2, \frac{1}{7})$ and the origin are on the same side of the line $3x + 7y - 8 = 0$.

69. Distance from a point to a line. A numerical example will be first worked through. Let it be required to find the distance from $(6, -3)$ to the line $3x - 5y = 7$.

Transform to parallel axes through the given point $(6, -3)$, as a new origin, the equations of transformation for which are

$$x = x' + 6, \quad y = y' - 3.$$

Substituting these values in the equation of the line, it becomes

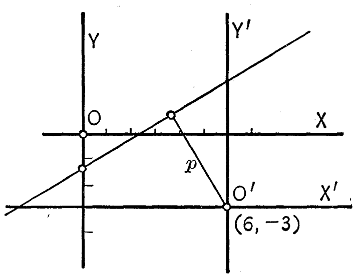


FIG. 71.

$$3x' - 5y' + 26 = 0,$$

which is the equation of the line referred to the new axes.

The distance from the new origin to the line is given by the formula

$$p = \frac{C}{\pm\sqrt{A^2+B^2}}, \qquad \text{(Art. 63)},$$

which here becomes

$$p = \frac{26}{\sqrt{34}} = 4.46 \text{ nearly}.$$

70. General formula for the distance from a point to a line. Let the given point be $P_0(x_0, y_0)$, and the given line

$$Ax + By + C = 0.$$

Transform to parallel axes through P_0 as a new origin, for which the formulas of transformation are

$$x = x_0 + x', \quad y = y_0 + y'.$$

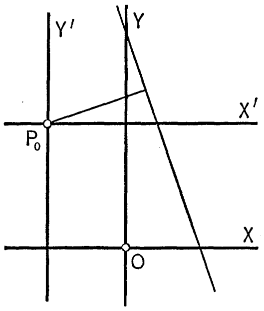


FIG. 72.

The equation of the line referred to the new axes is then

$$A(x' + x_0) + B(y' + y_0) + C = 0,$$

or $$Ax' + By' + Ax_0 + By_0 + C = 0.$$

In this equation x' and y' are the variable coördinates, and the constant term of the equation is $Ax_0 + By_0 + C$. Therefore the distance, d, from the new origin to the line is

$$\boldsymbol{d = \frac{Ax_0 + By_0 + C}{\pm\sqrt{A^2+B^2}}}.$$

This distance will be regarded as a positive quantity. The sign of the radical must therefore be taken the same as the sign of $Ax_0 + By_0 + C$. But $Ax_0 + By_0 + C$ has the same sign as C when (x_0, y_0) and the origin are on the same side of the line $Ax + By + C = 0$, and has the opposite sign to C when (x_0, y_0) and the origin are on opposite sides of the line (Art. 67).

Therefore the sign to be taken with the radical is the same as the sign of C when (x_0, y_0) and the origin are on the same side of $Ax + By + C = 0$, and opposite to the sign of C when (x_0, y_0) and the origin are on opposite sides of the line.

If $C = 0$, the sign to be taken with the radical is the same as or opposite to the sign of A according as (x_0, y_0) lies to the right or left of the line $Ax + By = 0$.

EXERCISE XX

1. Find the distance from the point $(3, -5)$ to the line $7x - 5y = 13$.

2. Find the distance from the intersection of $2x - 9y = 3$ and $-5y - 4x = 12$ to $x - 5 = 6y$.

3. The vertices of a triangle are $A(1, 4)$, $B(-3, -5)$, and $C(6, -4)$; find the area by finding the lengths of AB and the altitude from C to AB. Check by using the formula for the area of a triangle. (See Art. 36.)

4. The equations of the sides of a triangle are $x + 4y - 7 = 0$, $3x + y + 1 = 0$, and $2y - 5x + 13 = 0$; find the area of the triangle by finding the length of one side and the length of the perpendicular from the opposite vertex to that side. Check by using the formula for the area of a triangle. (See Art. 36.)

5. Find the distance from the intersection of $2x - 5y = 3$ and $8x + y + 13 = 0$ to the line through $(-\frac{3}{2}, 4)$ and $(0, -3)$.

6. Find the distance from $(9, -1)$ to the line through the origin with slope $-\frac{1}{2}$.

7. Find the distance from (x_1, y_1) to $y = mx + b$.

8. Find the distance from (x_0, y_0) to $x \cos \alpha + y \sin \alpha = p$.

9. Find the equations of the bisectors of the angles formed by the two lines $2x + y - 7 = 0$ and $4x - 3y - 5 = 0$. Show by their slopes that the bisectors are perpendicular to each other.

SUGGESTION. The distances from (x_0, y_0) to $2x + y - 7 = 0$ and $4x - 3y - 5 = 0$ are, respectively,

$$\frac{2x_0 + y_0 - 7}{\pm\sqrt{5}} \text{ and } \frac{4x_0 - 3y_0 - 5}{\pm\sqrt{25}}.$$

Now the bisector of an angle is the locus of points equidistant from the sides of the angle. Hence to obtain the equation of the bisector, place

the above values for the distances equal to each other and remove the subscripts to indicate variable coördinates. The proper signs of the radicals must be chosen, as explained in Art. 70.

10. The three sides of a triangle have the equations $3x - 4y = 7$, $5x + 12y + 8 = 0$, and $4x + 3y - 12 = 0$; find the equations of the three inner bisectors of the angles, and show by their equations that they meet in a point.

11. Find the equations of the three outer bisectors of the angles of the triangle of example 10, and prove by their equations that two of the outer bisectors and the inner bisector of the remaining angle meet in a common point.

71. Equations of the straight line in polar coördinates.

(i) **Equation of the straight line through two points.** Let $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ be any two points in the plane. To find the equation of the straight line passing through them. Let $P(r, \theta)$ be a point on the line as shown in Fig. 73.

Then $$\text{area } OP_1P_2 = \text{area } OP_1P + \text{area } OPP_2.$$

I.e. $$\tfrac{1}{2} r_1r_2 \sin(\theta_2 - \theta_1) = \tfrac{1}{2} rr_1 \sin(\theta - \theta_1) + \tfrac{1}{2} rr_2 \sin(\theta_2 - \theta),$$

or $$r_1r_2 \sin(\theta_2 - \theta_1) + r_2r \sin(\theta - \theta_2) + rr_1 \sin(\theta_1 - \theta) = 0.$$

This equation holds for any position of P on the line between P_1 and P_2. If P be so taken that P_2 lies between P and P_1, the equation that holds can be obtained from the above equation by interchanging r and r_2, and θ and θ_2. But this interchange only changes the sign of the left member of the equation, and since the right member is zero, the equation itself is unchanged. If P be so taken that P_1 lies between P and P_2, it may be shown in the same way that the same equation holds. Hence the above equation holds for all points on the line, and clearly for no others, and is therefore the equation of the line.

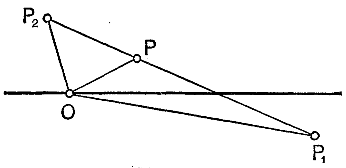


FIG. 73.

The same equation may be derived at once by equating to zero the area of the triangle whose vertices are P, P_1, and P_2. (See Art. 37.)

(ii) **Equation of a straight line in terms of the length of the perpendicular from the origin to the line and the angle which this perpendicular makes with the initial line.**

Let the perpendicular from the origin to the line be of length p, and make an angle α with the initial line. Let $P(r, \theta)$ be a point on the line. Then (Fig. 74) $r \cos(\alpha - \theta) = p$. Since $\cos(-A) = \cos A$, this may be written

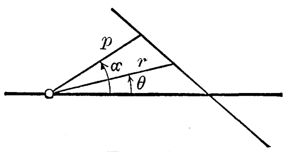


FIG. 74.

$$r \cos(\theta - \alpha) = p.$$

The student should show that this equation holds for all points on the line.

EXERCISE XXI

1. Write the equation of the line through $(2, 30°)$ and $(1, 60°)$.

2. Draw the line whose equation is $r \cos(\theta - 60°) = 5$.

3. Draw the line whose equation is $r \sin \theta = 4$.

4. Find the intersection of the lines $r \cos \theta = 8$, and $r \sin \theta = 4$.

5. Find the intersection of the lines $r \cos[\theta - \sin^{-1}(\frac{3}{5})] = 2$, and $r \cos[\theta - \cos^{-1}(\frac{5}{13})] = 4$.

6. Derive the equation $r \cos(\theta - \alpha) = p$ by substituting in $x \cos \alpha + y \sin \alpha = p$, the values $x = r \cos \theta$, $y = r \sin \theta$.

7. Derive the equation of the straight line through two points in polar coördinates by substituting in

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1},$$

the values $x = r \cos \theta$, $y = r \sin \theta$.

CHAPTER VII

STANDARD EQUATIONS OF SECOND DEGREE

CIRCLE, PARABOLA, ELLIPSE, HYPERBOLA

72. The circle. The equation of a circle of radius r and center at (h, k) is

$$(\boldsymbol{x}-\boldsymbol{h})^2+(\boldsymbol{y}-\boldsymbol{k})^2=\boldsymbol{r}^2.$$

PROOF. Denote the center by C. Let $P(x, y)$ be any point on the circle. The condition that P is on the circle is expressed by the equation

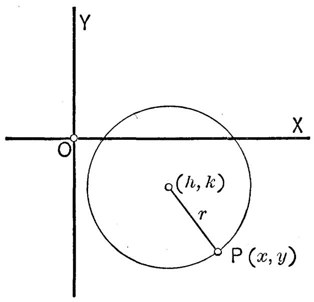


FIG. 75.

$$CP=r.$$

In terms of the coördinates of the points it becomes

$$(x-h)^2+(y-k)^2=r^2,$$

which is therefore the equation of the circle, for it is an equation which is satisfied by all points on the circle and by no others.

If the center is at the origin, the equation reduces to

$$\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{r}^2.$$

73. The equation $\quad \boldsymbol{x}^2+\boldsymbol{y}^2+\boldsymbol{D}\boldsymbol{x}+\boldsymbol{E}\boldsymbol{y}+\boldsymbol{F}=\mathbf{0}. \qquad (1)$

An equation of this form, by completing the square in the terms containing x and in those containing y, can be thrown

into the form of the equation of the preceding article as follows:

Add $\frac{D^2}{4}+\frac{E^2}{4}-F$ to both sides of eq. (1). The result is

$$x^2+Dx+\frac{D^2}{4}+y^2+Ey+\frac{E^2}{4}=\frac{D^2}{4}+\frac{E^2}{4}-F,$$

or

$$\left(x+\frac{D}{2}\right)^2+\left(y+\frac{E}{2}\right)^2=\frac{D^2+E^2-4F}{4}. \tag{2}$$

Now, if D^2+E^2-4F is positive, eq. (2) is, by the preceding article, the equation of a circle with center at $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ and radius equal to $\frac{1}{2}\sqrt{D^2+E^2-4F}$.

If $D^2+E^2-4F=0$, eq. (2), and hence eq. (1), is satisfied by $x=-\frac{D}{2}$, $y=-\frac{E}{2}$ only; for the sum of two terms, neither of which is negative, can vanish only when the terms vanish separately.

If $D^2+E^2-4F<0$, eq. (2), and hence eq. (1), is satisfied by no real values of x and y: for the sum of two quantities, neither of which is negative, cannot equal a negative quantity.

Hence the equation

$$x^2+y^2+Dx+Ey+F=0$$

represents

(1) a circle, center at $\left(-\frac{D}{2},-\frac{E}{2}\right)$, radius $=\frac{1}{2}\sqrt{D^2+E^2-4F}$,

if $D^2+E^2-4F>0$,

(2) a point $\left(-\frac{D}{2},-\frac{E}{2}\right)$, if $D^2+E^2-4F=0$,

(3) no locus, if $D^2+E^2-4F<0$.

74. The equation of a circle through three points. The equation of a circle through three given points, not in the

same straight line, can be found by use of the equation of the preceding article as is illustrated by the following example:

Example. To find the equation of a circle through (2, 1), (− 1, 3), and (− 3, − 4).

The equation of the required circle is of the form

$$x^2 + y^2 + Dx + Ey + F = 0. \tag{1}$$

The coördinates of each of the given points must satisfy this equation. Therefore

$$\begin{aligned} 5 + 2D + E + F &= 0, \\ 10 - D + 3E + F &= 0, \\ 25 - 3D - 4E + F &= 0. \end{aligned}$$

The values of D, E, and F, obtained by solving these equation, are $D = \frac{13}{5}$, $E = \frac{7}{5}$, $F = -\frac{58}{5}$.

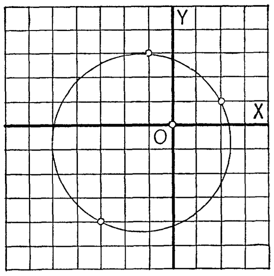


Fig. 76.

Substituting these values in eq. (1) and clearing of fractions, the required equation of the circle is

$$5(x^2 + y^2) + 13x + 7y - 58 = 0.$$

Using the formulas of Art. 73, the center and radius of the circle are found to be $(-1.3, -.7)$ and $r = 3.71. \ldots$

A check on the work is obtained by plotting the points and drawing the circle with center and radius as computed. (Fig. 76.)

EXERCISE XXII

Find the centers and radii of the circles represented by the following equations. Draw the figures.

1. $x^2 + y^2 = 25$.
2. $x^2 + y^2 - 4x + 6y = 12$.
3. $x^2 + y^2 + 8x - 6y = 0$.
4. $2x^2 + 2y^2 - 7y + 3x = 11$.
5. $(x - 1)^2 + (y + 2)^2 = 0$.
6. $(x - h)^2 + (y - k)^2 = 0$.
7. $u^2 + v^2 + u + v = 0$.
8. $x^2 + y^2 - 4x + 6y + 14 = 0$.

9. $x^2 + y^2 - 4x + 6y - 13 = 0$. 10. $x^2 - 2ax + y^2 = 0$.

11. $x^2 - 2ax + y^2 - 2ay = 0$. 12. $x^2 + y^2 - ax - by = 0$.

Find the equations of circles which fulfill the following conditions:

13. Center at $(-1, 3)$, radius $= 2$.

14. Center at $(a, 0)$, radius $= a$.

15. Center at the intersection of $y + 4x + 1 = 0$ and $2x - y + 5 = 0$, and passing through $(2, -3)$.

16. Center at $(2, 5)$ and tangent to $3x + 4y = 11$.

17. Center on the line $y = 2x$ and passing through $(0, 5)$ and $(6, 1)$.

18. Passing through $(0, 2)$, $(-1, 3)$, and $(5, 0)$.

19. Circumscribing the triangle whose sides are $5x + 3y - 14 = 0$, $4x - 3y + 5 = 0$, and $x + 6y + 8 = 0$.

20. Inscribed in the triangle whose sides are $5x + 12y - 2 = 0$, $4x + 3y + 5 = 0$, and $3x - 4y - 15 = 0$.

21. Tangent internally to the first two sides of the triangle mentioned in example 20, and tangent externally to the other side.

22. Prove that if $P_1(x_1, y_1)$ is any point without the circle

$$(x - h)^2 + (y - k)^2 - r^2 = 0,$$

and T is the point of contact of a tangent drawn from P_1 to the circle,

then, $$\overline{P_1T}^2 = (x_1 - h)^2 + (y_1 - k)^2 - r^2.$$

23. Show that if the equation of the circle of example 22 is

$$x^2 + y^2 + Dx + Ey + F = 0,$$

then, $$\overline{P_1T}^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F.$$

24. Prove that the locus of points from which equal tangents may be drawn to

$$x^2 + y^2 + D_1x + E_1y + F_1 = 0,$$

and $$x^2 + y^2 + D_2x + E_2y + F_2 = 0,$$

is the straight line

$$(D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0,$$

or, in case the circles intersect, is that portion of the line not inside of the circles.

This line is called the **radical axis** of the two circles.

25. Show that if two circles intersect, their radical axis passes through their points of intersection.

26. Find the equations of the radical axes of the circles of examples 1, 2, and 3, and prove that they meet in a point.

27. Prove that the three radical axes of any three circles taken in pairs meet in a common point.

28. Prove that the radical axis of two circles is perpendicular to their line of centers.

75. The parabola. The **parabola** is the locus of a point which moves so as to keep equidistant from a fixed point and a fixed straight line.

The fixed point is called the **focus**, the fixed straight line the **directrix**, of the parabola.

To obtain the equation of the parabola, let, at first, the directrix be taken as the axis of y and the focus at the point $(p, 0)$.

Let $P(x, y)$ be any point on the locus. Join P and $F(p, 0)$, and draw MP parallel to the x-axis to meet the y-axis in M. Then the condition that P is the point on the locus is expressed by the equation

$FP = MP$, if MP is positive, and by
$FP = -MP$, if MP is negative.

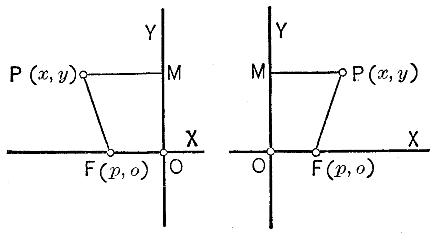


FIG. 77.

Evidently MP is positive or negative according as $(p, 0)$ lies to the right or the left of the origin, *i.e.* according as p is positive or negative.

Now $$FP = \sqrt{(x-p)^2 + y^2},$$
and $$MP = x.$$

$$\therefore \quad \sqrt{(x-p)^2 + y^2} = x, \text{ if } p > 0, \tag{1}$$

and $$\sqrt{(x-p)^2 + y^2} = -x, \text{ if } p < 0. \tag{2}$$

Squaring and transposing, either of these equations becomes

$$y^2 = 2\,px - p^2. \tag{3)*$$

This equation may be written

$$y^2 = 2\,p\left(x - \frac{p}{2}\right).$$

Let the axes be translated to $\left(\frac{p}{2}, 0\right)$ as a new origin. The formulas of transformation are

$$x = x' + \frac{p}{2},\ y = y'.$$

The equation of the parabola referred to the new axes is, therefore,

$$y'^2 = 2\,px'. \tag{4}$$

Dropping primes,

$$\boldsymbol{y^2 = 2\,px} \tag{5}$$

is therefore the equation of a parabola when the y-axis is parallel to the directrix through a point halfway between the focus and directrix, the x-axis passes through the focus and is perpendicular to the directrix, and the focus is at $\left(\frac{p}{2}, 0\right)$.

It is important to note that in eq. (5) the abscissas of points on the parabola vary as the square of the ordinates.

76. The graph of $y^2 = 2\,px$.

I. p positive.

* Equation (3) is not equivalent to both eqs. (1) and (2), but only to (1) if p is positive, and to (2) if p is negative. For on retracing the steps from (3) the eq. $\sqrt{(x-p)^2 + y^2} = \pm\, x$ is obtained. Now the + sign can only be used when x is positive, since the radical is counted positive. But if p were negative when x is positive, then $\sqrt{(x-p)^2 + y^2}$ would be greater than x. $\therefore$ when x is positive, p is positive. Therefore the + sign of x can be taken only when p is positive. Hence when p is positive, eq. (3) is satisfied by precisely the same points as eq. (1).

In the same way it can be shown that eq. (3) is equivalent to eq. (2) when p is negative.

(1) The curve is symmetric with respect to the x-axis.

(2) When $x = 0, y = 0$; when $y = 0, x = 0$. The curve therefore meets the axes at (0, 0) only.

(3) All negative values of x make y imaginary. The curve, therefore, lies to the right of the y-axis.

(4) No finite value of either variable makes the other infinite.

(5) As x increases, the positive value of y increases, a small change in x making a small change in y.

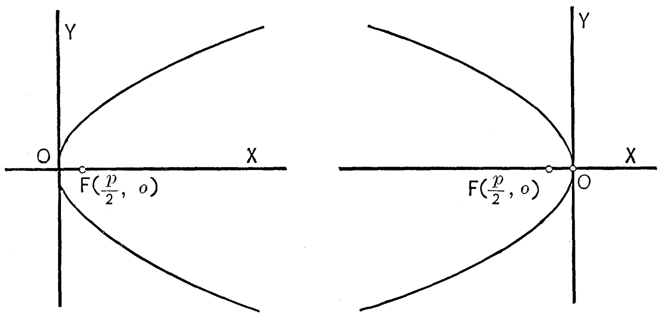
(6) When x becomes infinite, y also becomes infinite.

The upper half of the curve may, therefore, be generated by a point which, starting at (0, 0), moves ever to the right and upward, receding indefinitely from both axes.

The following points are on the curve:

x	0	$\frac{p}{8}$	$\frac{p}{2}$	p	$2p$	$3p$	$4p$	$8p$	$50p$	$200p$,
y	0	$\pm\frac{p}{2}$	$\pm p$	$\pm p\sqrt{2}$	$\pm 2p$	$\pm p\sqrt{6}$	$\pm 2p\sqrt{2}$	$\pm 4p$	$\pm 10p$	$\pm 20p$.

The curve is shown in Fig. 78 for a certain value of p.



p positive $y^2 = 2px$ p negative

FIG. 78.

II. p negative. This case differs from that in which p is positive only in making the curve lie to the left of the y-axis instead of the right. The curve is shown in Fig. 78, the values

of p for the two curves being numerically equal, but opposite in sign.

77. Axis of a parabola. Vertex. The straight line through the focus perpendicular to the directrix is called the **axis** of the parabola.

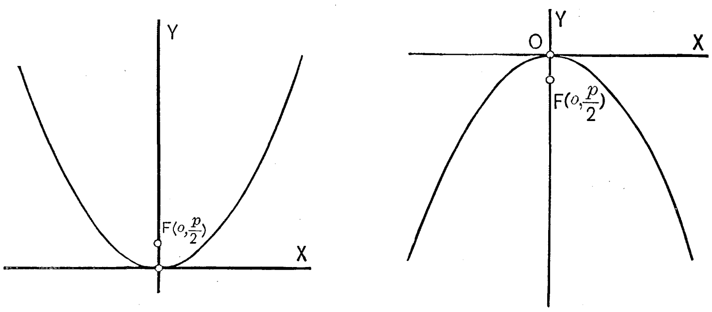
The point where the parabola crosses its axis is called the **vertex**.

In both cases of Fig. 78 the x-axis is the axis of the parabola, and the vertex is at the origin.

78. Parabola with axis on the y-axis and vertex at the origin. If the vertex is at the origin and the focus at $\left(0, \frac{p}{2}\right)$, the equation can evidently be obtained from that of Art. 76 by exchanging x and y. The equation is therefore

$$x^2 = 2py.$$

The two cases are shown in Fig. 79.



p positive. $\qquad$ $x^2 = 2py.$ $\qquad$ p negative.

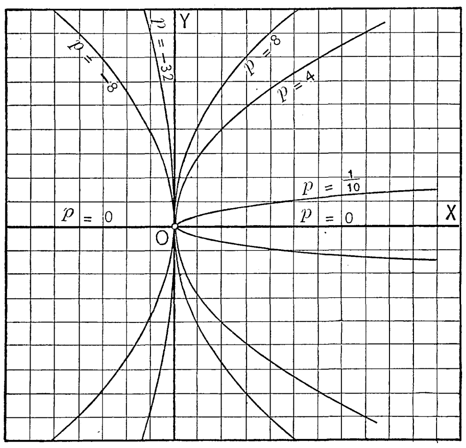
Fig. 79.

79. The arbitrary constant of the parabola.

DEFINITION. A constant which may have any value in an equation is called an **arbitrary constant**, or a **parameter**, of the equation.

In the equation of the parabola, $y^2 = 2px$, since p may have any real value, it is an arbitrary constant of the equation.

Corresponding to each value of p there is a definite curve. The curves which correspond to a few different values of p are sketched in Fig. 80.



$y^2 = 2px$.

FIG. 80.

80. The equations

$$Ax^2 + Dx + Ey + F = 0, \quad A \neq 0, \quad E \neq 0$$
$$Cy^2 + Dx + Ey + F = 0, \quad C \neq 0, \quad D \neq 0.$$

Equations of these forms can by a translation of axes be thrown into the forms $\qquad x^2 = 2py$

and $\qquad y^2 = 2px$

respectively. The equations therefore represent parabolas with their axes parallel respectively to the y-axis and the x-axis.

A numerical example will make this clear.

EXAMPLE. $$3x^2 + 2x + 5y - 4 = 0.$$

Complete the square in the terms containing x, and transpose the other terms to the other side of the equation:

$$3\left(x^2 + \tfrac{2}{3}x + \tfrac{1}{9}\right) = -5y + 4 + \tfrac{1}{3} = -5y + \tfrac{13}{3},$$

or $$\left(x + \tfrac{1}{3}\right)^2 = -\tfrac{5}{3}\left(y - \tfrac{13}{15}\right).$$

Translate the axes to $(-\frac{1}{3}, \frac{13}{15})$ as a new origin by means of the equations

$$x = x' - \tfrac{1}{3}, \quad y = y' + \tfrac{13}{15}.$$

The transformed equation is

$$x'^2 = -\tfrac{5}{3}y'.$$

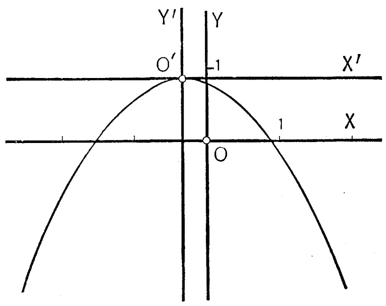


FIG. 81.

This is the equation of a parabola with axis on the new y-axis, vertex at the origin, and focus at $(0, -\frac{5}{6})$ referred to the new axes.

Referred to the old axes, the vertex is $(-\frac{1}{3}, \frac{13}{15})$, the focus is $(-\frac{1}{3}, \frac{1}{30})$, and the equation of the axis of the parabola is $x = -\frac{1}{3}$.

81. The equation $\boldsymbol{y = ax^2 + bx + c}$.

This equation represents a parabola with its axis parallel to

the y-axis. For, it may be written

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a},$$

or

$$y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2.$$

Let

$$x' = x + \frac{b}{2a},$$

$$y' = y + \frac{b^2 - 4ac}{4a},$$

and this equation becomes

$$x'^2 = \frac{1}{a}y',$$

which is the equation of a parabola with axis on the new y-axis and vertex at the new origin. Hence, referred to the old axes the vertex is at $\left(-\frac{b}{2a}, \quad -\frac{b^2 - 4ac}{4a}\right)$, and the axis of the parabola is parallel to the y-axis.

The parabola extends upward or downward from the vertex according as a is positive or negative. The sign of $b^2 - 4ac$ determines whether or not the curve crosses the x-axis.

Let the student show that the conditions are as stated in Fig. 82.

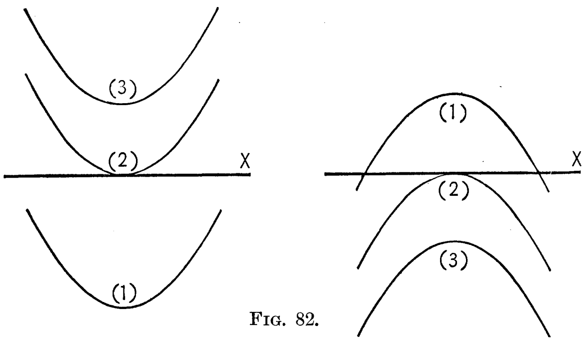


FIG. 82.

a positive $\quad y = ax^2 + bx + c. \quad$ a negative.

(1) $b^2 - 4ac > 0$. (2) $b^2 - 4ac = 0$. (3) $b^2 - 4ac < 0$.

82. The parabolic arch. An arch of height h and span $2l$ is in the form of a parabola with vertex at the crown. It is desirable to compute readily the heights of the arch at varying distances from the center of the span.

Choose the axes as in Fig. 83, counting y as positive down-

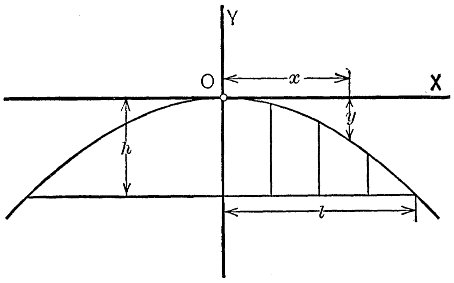


FIG. 83.

ward. Since in a parabola, and with this choice of axes, the ordinate varies as the square of the abscissa, therefore

$$\frac{y}{h} = \frac{x^2}{l^2}.$$

This form of the equation enables one to compute readily the heights at varying distances from the center.

E.g. the heights at distances from the center of $\frac{l}{4}$, $\frac{l}{2}$, and $\frac{3l}{4}$ are respectively $\frac{15h}{16}$ $\frac{3h}{4}$, and $\frac{7h}{16}$.

EXERCISE XXIII

Plot the following parabolas, finding the vertex of each and reducing the equation to the standard form. In each case compute the value of the discriminant, $b^2 - 4ac$.

1. $y = 2x^2 - 3x + 5$.
2. $y = -3x^2 + 4x - 1$.
3. $y = x^2 + 4x + 4$.
4. $5y - 2x^2 + 4x - 3 = 0$.
5. $2x^2 + 3x + y = 4$.
6. $3y^2 - 2y + 4x - 3 = 0$.

7. Discuss the effect upon the position and form of the curve caused by separately varying the quantities a, b, and c in the equation

$$y = ax^2 + bx + c.$$

8. Discuss the equation $x = ay^2 + by + c$. (Compare Art. 81.)

9. A parabolic arch of 60 ft. span is 20 ft. high at the center. Compute the heights at intervals of 5 ft. from the center.

10. Through how many arbitrarily assigned points can a parabola with axis parallel to one of the coördinate axes be passed in general? Name some exceptions.

Find the equation of a parabola with axis parallel to the y-axis through the three points $(1, 0)$, $(3, 2)$, and $(6, 8)$. Draw the figure.

11. Find the equation of a parabola through the three points of example 10 with axis parallel to the x-axis.

12. Find the equation of a parabola through the points $(-h, y_1)$, $(0, y_2)$, and (h, y_3) with axis parallel to the y-axis.

13. Find the equation of a parabola through $(1, 0)$, $(3, 2)$, and $(6, 5)$.

14. Find the equation of a parabola with axis parallel to the y-axis passing through $(-20, 0)$, $(0, \frac{1}{10})$, and $(20, 0)$.

Can a parabola with axis parallel to the x-axis be passed through these points?

15. Show that any line parallel to the axis of a parabola cuts the parabola in one and only one point.

83. The ellipse. The **ellipse** is the locus of a point which moves in the plane so that the sum of its distances from two fixed points in the plane is constant.

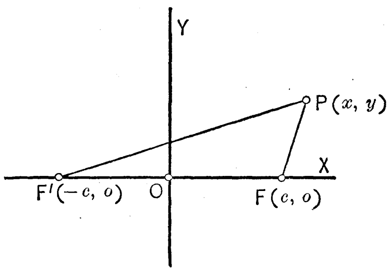


FIG. 84.

The fixed points are called the **foci**.

To obtain the equation of the ellipse, let the x-axis be taken through the foci, and the origin midway between the foci. Let the distance between the foci be $2c$. The foci are then $F(c, 0)$ and $F'(-c, 0)$.

Call the given constant $2a$, where $2a > 2c$.

Let $P(x, y)$ be any point of the ellipse; then

$$\boldsymbol{FP + F'P = 2a}.$$

In terms of the coördinates of the points, this becomes

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a. \tag{1}$$

This is therefore the equation of the ellipse.

Equation (1) can be thrown into a more convenient form free from radicals as follows: Transpose the second radical to the right-hand member of the equation and square,

$$x^2-2cx+c^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2+y^2. \tag{2}$$

Canceling, transposing, and dividing by 4,

$$cx+a^2=a\sqrt{(x+c)^2+y^2}. \tag{3}$$

Squaring,

$$c^2x^2+2a^2cx+a^4=a^2x^2+2a^2cx+a^2c^2+a^2y^2. \tag{4}$$

Canceling and collecting terms,

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2). \tag{5}$$

Dividing by $a^2(a^2-c^2)$,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1. \tag{6}$$

All values of x and y that satisfy eq. (1) also satisfy eq. (6), but in obtaining (6) from (1) the operation of squaring was twice performed, and in this process there are introduced values of x and y which satisfy eq. (6) but do not satisfy eq. (1). However, the values so introduced in this case are imaginary, and hence there are no points on the locus of eq. (6) that are not also on the locus of eq. (1). For, starting with eq. (6), the steps may be retraced until eq. (4) is reached, where, upon extracting the square root, a double sign is introduced, *i.e.*

$$cx+a^2=\pm a\sqrt{(x+c)^2+y^2},$$

or

$$-cx=a^2\mp a\sqrt{(x+c)^2+y^2}. \tag{3'}$$

Multiply by 4 and add $(x+c)^2+y^2$ to both sides,

$$x^2+2cx+c^2+y^2-4cx=4a^2\mp 4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2,$$

or
$$(x-c)^2+y^2=(2a\mp\sqrt{(x+c)^2+y^2})^2. \qquad (2')$$

Extract the square root,

$$\pm\sqrt{(x-c)^2+y^2}=2a\mp\sqrt{(x+c)^2+y^2},$$

or
$$\pm\sqrt{(x-c)^2+y^2}\pm\sqrt{(x+c)^2+y^2}=2a. \qquad (1')$$

Therefore, if (x, y) is denoted by P,

$$\pm PF \pm PF' = 2a.$$

Now $2a$ is a positive quantity; hence both negative signs cannot be used. Also $FF'=2c$. The difference of PF' and PF is therefore less than $2c$. (Fig. 84.) That difference cannot therefore be equal to $2a$, which is greater than $2c$. Hence the only allowable combination of signs for real values of x and y is given in

$$+PF+PF'=2a.$$

Therefore all real values of x and y that satisfy eq. (6) also satisfy eq. (1). Hence eq. (6) is the equation of the ellipse.

Replacing the positive quantity $\boldsymbol{a^2-c^2}$ in eq. (6) by $\boldsymbol{b^2}$, the equation becomes

$$\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.}$$

84. Graph of $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.}$

Solving for y, $$y=\pm\frac{b}{a}\sqrt{a^2-x^2}.$$

Solving for x, $$x=\pm\frac{a}{b}\sqrt{b^2-y^2}.$$

(1) The curve is symmetric with respect to both coördinate axes, and the origin.

(2) It crosses the x-axis at $(a, 0)$ and $(-a, 0)$ and the y-axis at $(0, b)$ and $(0, -b)$.

(3) If x is less than $-a$ or greater than a, y is imaginary. If y is less than $-b$ or greater than b, x is imaginary. Therefore no portion of the locus lies to the left of $x=-a$ or to the right of $x=a$; below $y=-b$, or above $y=b$.

(4) No finite value of either variable makes the other infinite.

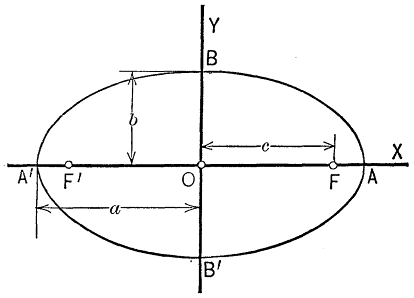
(5) In the first quadrant, as x increases from 0 to a, y steadily decreases from b to 0.

(6) By (3) neither variable can become infinite.

The following points are on the curve,

x	0	$\frac{a}{4}$	$\frac{a}{2}$	$\frac{3a}{4}$	$\frac{7a}{8}$	$\frac{9a}{10}$	a
y	b	$.97b$	$.87b$	$.66b$	$.48b$	$.44b$	0

The curve is sketched in Fig. 85.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

FIG. 85.

85. Axes, vertices, center of the ellipse.

DEFINITIONS. The chord of the ellipse which passes through the foci is called the **major axis** of the ellipse; the chord at right angles to the major axis and passing through its center, the **minor axis**; their intersection the **center**, and the ends of the major axis the **vertices** of the ellipse.

Thus in Fig. 85, $A'A=2a$ is the major axis, $B'B=2b$ is the minor axis, O is the center, $A(a, 0)$ and $A'(-a, 0)$ are the vertices.

In terms of a and b the foci are $F(\sqrt{a^2-b^2},\ 0)$ and $F'(-\sqrt{a^2-b^2},\ 0)$,

since $$b^2=a^2-c^2 \text{ or } c=\sqrt{a^2-b^2}.$$

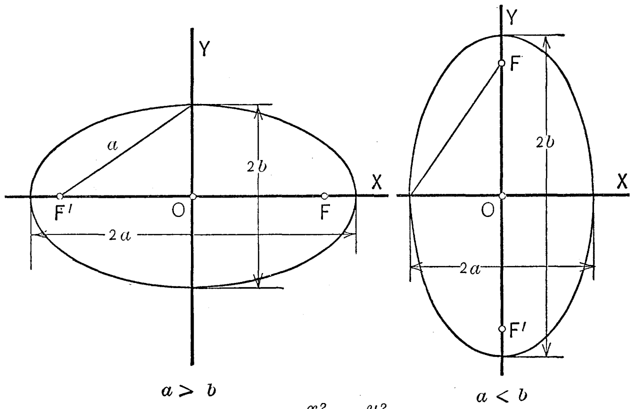
86. The ellipse with major axis on the y-axis. In Art. 83 the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ was found for the ellipse with center at the origin and with major axis $2a$ on the x-axis. If the major axis $2a$ were taken on the y-axis, the equation would clearly be obtained from that above by exchanging x and y. It is therefore,

$$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1, \text{ where the major axis is } 2a.$$

If, however, the major axis is called $2b$ and the minor axis $2a$, the equation becomes the same as that of Art. 83; namely,

$$\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.}$$

This equation therefore represents an ellipse with major axis on the x-axis or the y-axis according as a is greater than



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

FIG. 86.

or less than b. In the latter case the foci are $(0, \sqrt{b^2-a^2})$ and $(0, -\sqrt{b^2-a^2})$.

87. The hyperbola. An **hyperbola** is the locus of a point which moves in the plane so that the difference of its distances from two fixed points of the plane is constant.

To find the equation of the hyperbola, as in Art. 83, let the fixed points be $F(c, 0)$ and $F'(-c, 0)$, and let the constant be $2a$. Here, however, $2a < 2c$, since the difference between two sides of a triangle is less than the other side.

Let $P(x, y)$ be any point of the locus, then (Fig. 87) either

$$FP - F'P = 2a \text{ or } F'P - FP = 2a,$$

according as P is nearer to F' or F.

$$FP - F'P = \pm 2a.$$

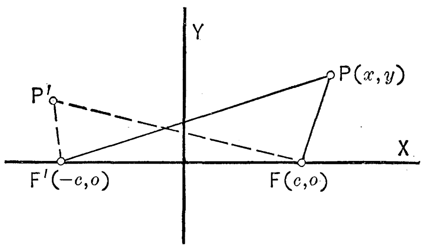


Fig. 87.

Expressed in terms of the coördinates, this equation becomes

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a. \qquad (1)$$

This is therefore the equation of the hyperbola.

It is more convenient to have the equation free from radicals.

Transposing the second term and squaring,

$$x^2-2cx+c^2+y^2=x^2+2cx+c^2+y^2\pm 4a\sqrt{(x+c)^2+y^2}+4a^2. \qquad (2)$$

Canceling, transposing, and dividing by 4,

$$-(cx+a^2) = \pm a\sqrt{(x+c)^2+y^2}. \qquad (3)$$

Squaring,

$$c^2x^2 + 2\,a^2cx + a^4 = a^2x^2 + 2\,a^2cx + a^2c^2 + a^2y^2. \qquad (4)$$

Canceling and collecting terms,

$$(c^2 - a^2)\,x^2 - a^2y^2 = a^2(c^2 - a^2). \qquad (5)$$

Dividing by $a^2(c^2 - a^2)$,

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1. \qquad (6)$$

This is precisely the same form as eq. (6) in Art. 83, the only difference being that here $c^2 - a^2$ is positive, whereas there it was negative.

Every point whose coördinates satisfy eq. (1) also satisfy eq. (6). That conversely all points which satisfy eq. (6) also satisfy eq. (1) may be shown as in Art. 83. The steps by which (6) was obtained from (1) can be retraced, but a double sign must be used when the square root is extracted. Hence, given eq. (6), there follows

$$\pm\sqrt{(x - c)^2 + y^2} \pm \sqrt{(x + c)^2 + y^2} = \pm\, 2\,a. \qquad (1')$$

If $P(x, y)$ is any point on the locus of this equation, then

$$\pm\, FP \pm F'P = \pm\, 2\,a.$$

The same sign cannot be used throughout, since the sum of two sides of a triangle is greater than the third side, and $2\,a < 2\,c$.

The same signs for the terms on the left and the opposite sign on the right cannot be used, since the sum of two positive quantities is positive.

Hence the only combinations of signs left is that where the signs of the terms on the left are different, which is equivalent to

$$FP - F'P = \pm\, 2\,a.$$

Therefore eq. (6) is satisfied by only those points which satisfy eq. (1). Equation (6) is therefore the equation of the hyperbola.

Letting the positive quantity $\boldsymbol{c^2 - a^2 = b^2}$, eq. (6) becomes

$$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} - \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} = \mathbf{1}.$$

88. Graph of $\dfrac{\boldsymbol{x^2}}{\boldsymbol{a^2}} - \dfrac{\boldsymbol{y^2}}{\boldsymbol{b^2}} = \mathbf{1}.$

Solving for y, $$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}.$$

Solving for x, $$x = \pm \frac{a}{b}\sqrt{y^2 + b^2}.$$

(1) The curve is symmetric with respect to both coördinate axes and the origin.

(2) It intersects the x-axis at $(a, 0)$ and $(-a, 0)$, but does not intersect the y-axis.

(3) If x lies between $-a$ and a, y is imaginary. All values of y make x real.

(4) No finite value of either variable makes the other infinite.

(5) In the first quadrant as x, starting at a, increases, y, starting at 0, steadily increases.

(6) As either variable becomes infinite, so does the other.

The part of the curve that lies in the first quadrant may therefore be generated by a point which, starting at $(a, 0)$, moves to the right and upward, receding indefinitely from both axes.

The following points are on the curve:

x	a	$\frac{3a}{2}$	$2a$	$3a$	$4a$	$10a$	$100a$,
y	0	$1.1b$	$1.7b$	$2.8b$	$3.9b$	$9.95b$	$99.99b$.

The curve is shown in Fig. 88.

The foci are $(\sqrt{a^2+b^2}, 0)$ and $(-\sqrt{a^2+b^2}, 0)$.

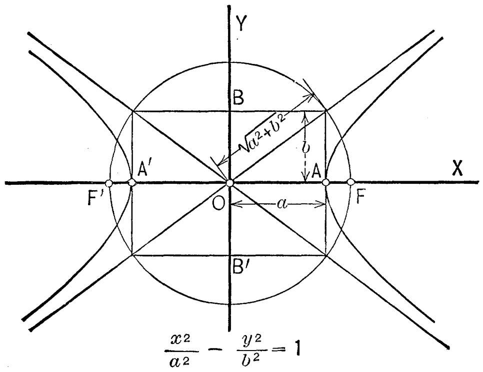


Fig. 88.

89. The asymptotes of the hyperbola. By observing the table of values of x and y in the preceding article, it may be seen that the ratio of x to y comes nearer and nearer to $\frac{a}{b}$ as x increases.

Consider then the locus of the equation $\frac{y}{x} = \frac{b}{a}$, or

$$\boldsymbol{y = \frac{b}{a} x.}$$

The locus of this equation is a straight line with slope $\frac{b}{a}$ and y-intercept 0 (Art. 55); *i.e.* it is a straight line through the origin and (a, b).

Let y_l and y_h denote the ordinates of the points on the line and the hyperbola respectively, in the first quadrant, for the same value of x. Form the difference

$$y_l - y_h = \frac{b}{a}(x - \sqrt{x^2 - a^2}),$$

which may be written

$$y_l - y_h = \frac{b\,(x^2 - x^2 + a^2)}{a\,(x + \sqrt{x^2 - a^2})} = \frac{ab}{x + \sqrt{x^2 - a^2}},$$

from which it is evident that, in the first quadrant, $y_l - y_h$ is positive, decreases as x increases, and approaches the limiting value 0 as x becomes infinite. The curve therefore comes ever nearer to the straight line as x increases, and approaches indefinitely near as x becomes infinite.

The line $\boldsymbol{y = \frac{b}{a} x}$ is therefore an **asymptote** of the curve.

From symmetry the same line is an asymptote in the third quadrant, and $\boldsymbol{y = -\frac{b}{a} x}$ is an asymptote in the second and fourth quadrants.

The asymptotes are shown in Fig. 88.

In plotting the hyperbola it is well to draw the asymptotes first. They will serve as an aid in sketching the curve when a very few points have been located.

90. Axes, vertices, center, of the hyperbola.

Definitions. The points of intersection of the hyperbola and the line through the foci are called the **vertices** of the hyperbola; the line joining the vertices the **transverse axis**; the middle point of this line the **center** of the hyperbola; and the line through the center perpendicular to the transverse axis, of length $2\sqrt{c^2 - a^2}$, the **conjugate axis.**

Thus in Fig. 88, A and A' are the vertices, $A'A = 2\,a$ is the transverse axis, the origin is the center, and $B'B = 2\,b$ is the conjugate axis.

91. The conjugate hyperbola. The equation of an hyperbola with foci on the y-axis at the points $(0, \sqrt{a^2 + b^2})$ and $(0, -\sqrt{a^2 + b^2})$ and transverse axis $2\,a$ is obtained from the equation of Art. 87 by interchanging x and y. The equation is therefore

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

If, however, the transverse axis is $2b$ and the conjugate axis $2a$, the equation is $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, or

$$\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.}$$

This hyperbola is called the **conjugate** of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

It is easily shown that the conjugate hyperbola also has the lines $y=\frac{bx}{a}$ and $y=-\frac{bx}{a}$ for asymptotes. The proof is left as an exercise.

The curve is shown in Fig. 89 together with the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

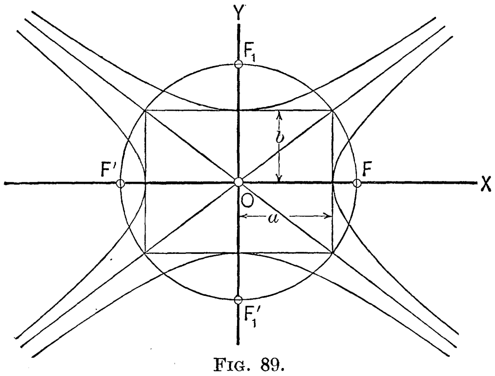


FIG. 89.

Since the equations of the asymptotes can be combined into the one equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$, therefore the three equations

$$\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,}$$

$$\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,}$$

$$\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=0,}$$

represent respectively an hyperbola, the conjugate hyperbola, and the asymptotes.

92. The equilateral hyperbola. If $b=a$, the hyperbola is called the **equilateral**, or **rectangular**, hyperbola.

The equation is

$$\boldsymbol{x^2-y^2=a^2}.$$

The asymptotes are $\boldsymbol{y=x}$ and $\boldsymbol{y=-x}$, and are therefore at right angles to each other.

93. The equilateral hyperbola referred to its asymptotes as axes. In the equation of the equilateral hyperbola of the preceding article, let the axes be rotated through an angle of $-45°$. The asymptotes then become the axes. The formulas of transformation are

$$x=x'\cos(-45°)-y'\sin(-45°),$$
$$y=x'\sin(-45°)+y'\cos(-45°),$$
$$x=\frac{1}{\sqrt{2}}(x'+y'),$$
$$y=\frac{1}{\sqrt{2}}(-x'+y').$$

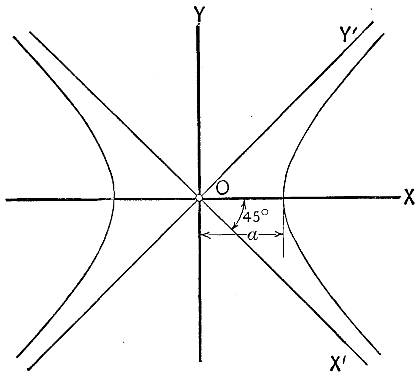


Fig. 90.

Substituting these values of x and y in the equation of the equilateral hyperbola

$$x^2-y^2=a^2,$$

there results

$$\tfrac{1}{2}(x'^2+2\,x'y'+y'^2)-\tfrac{1}{2}(x'^2-2\,x'y'+y'^2)=a^2,$$

or, dropping primes, $$\boldsymbol{xy=\frac{a^2}{2}}.$$

This is, therefore, the equation of the equilateral hyperbola referred to the asymptotes as axes.

From the above it follows that if two variables change in such a way that their product remains constant, the curve which represents the equation connecting them in rectangular coördinates is an equilateral hyperbola. *E.g.* the equation $pv=C$ is represented by an equilateral hyperbola.

94. The equation $\boldsymbol{Ax^2+Cy^2+Dx+Ey+F=0,\ A\neq 0,\ C\neq 0.}$

An equation of this form can, by a translation of axes, be transformed into one in which the terms of the first degree are lacking. For, completing the squares in the terms containing x and in those containing y, the equation becomes

$$A\left(x^2+\frac{D}{A}x+\frac{D^2}{4\,A^2}\right)+C\left(y^2+\frac{E}{C}y+\frac{E^2}{4\,C^2}\right)=\frac{D^2}{4\,A}+\frac{E^2}{4\,C}-F.$$

Letting $x=x'-\dfrac{D}{2\,A}$, $y=y'-\dfrac{E}{2\,C}$, and $\dfrac{D^2}{4\,A}+\dfrac{E^2}{4\,C}-F=F'$,

the equation becomes

$$Ax'^2+Cy'^2=F'.$$

The locus of this equation depends upon the values of A, C, and F'.

Suppose I. $F'=0$.

(1) Then, if A and C are of the same sign, no real values of x' and y' except $(0, 0)$ will satisfy the equation. Hence the locus is a point.

(2) If A and C are opposite in sign, $Ax'^2 + Cy'^2 = 0$ can be factored into two factors of first degree in x' and y', and therefore the locus is two intersecting straight lines.

II. $F \neq 0$. Divide by F',

$$\frac{x'^2}{\frac{F'}{A}} + \frac{y'^2}{\frac{F'}{C}} = 1.$$

(1) If $\frac{F'}{A}$ and $\frac{F'}{C}$ are both positive, the locus is an ellipse. (A circle if $A = C$.)

(2) If $\frac{F'}{A}$ and $\frac{F'}{C}$ are both negative, no real values of x' and y' satisfy the equation. Hence there is no locus.

(3) If $\frac{F'}{A}$ and $\frac{F'}{C}$ are opposite in sign, the locus is an hyperbola.

95. Illustrations.

EXAMPLE 1. $3x^2 - 4y^2 - 7x + 5y + 2 = 0$.
This may be written

$$3\left(x^2 - \frac{7x}{3} + \frac{49}{36}\right) - 4\left(y^2 - \frac{5y}{4} + \frac{25}{64}\right) = \frac{49}{12} - \frac{25}{16} - 2,$$

or
$$3(x - \tfrac{7}{6})^2 - 4(y - \tfrac{5}{8})^2 = \tfrac{25}{48}.$$

Let
$$x = x' + \tfrac{7}{6},\ y = y' + \tfrac{5}{8},$$

then
$$3x'^2 - 4y'^2 = \tfrac{25}{48},$$

or
$$\frac{x'^2}{\frac{25}{144}} - \frac{y'^2}{\frac{25}{192}} = 1.$$

This is the equation of an hyperbola with center at the new origin, transverse axis on the new x-axis, with $a = \frac{5}{12}$, $b = \frac{5\sqrt{3}}{24}$.

Referred to the old axes the center is at $(\frac{7}{6}, \frac{5}{8})$. (See Fig. 91.)

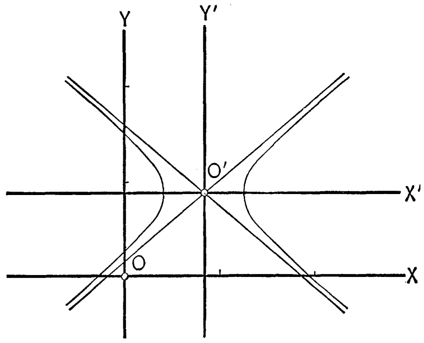


FIG. 91.

EXAMPLE 2. $x^2 - 9y^2 + 7x + 9y + 10 = 0.$

This may be written

$$x^2 + 7x + \tfrac{49}{4} - 9(y^2 - y + \tfrac{1}{4}) = \tfrac{49}{4} - \tfrac{9}{4} - 10,$$

or

$$(x + \tfrac{7}{2})^2 - 9(y - \tfrac{1}{2})^2 = 0.$$

Let

$$x = x' - \tfrac{7}{2},\ y = y' + \tfrac{1}{2}.$$

Then,

$$x'^2 - 9y'^2 = 0,$$

which may be written

$$(x' + 3y')(x' - 3y') = 0.$$

This equation is satisfied by all values of x' and y' that

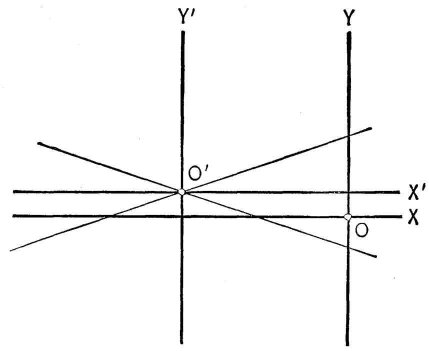


FIG. 92.

make either $x' - 3y' = 0$, or $x' + 3y' = 0$, and by no other values. The locus is therefore two straight lines through the new origin, with slopes $\frac{1}{3}$ and $-\frac{1}{3}$ respectively.

Referred to the old axes the point of intersection of the lines is $(-\frac{7}{2}, \frac{1}{2})$.

96. The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. An equation of this form can by a rotation of axes be reduced to one in which the term in xy is lacking. (Compare Art. 54, example 3.) The resulting equation can then be treated as in the preceding article.

Usually, where the xy-term and terms of the first degree appear in the equation, it is easier to first remove the terms of first degree by a translation of axes, and then remove the term in xy by a rotation of axes. It is not, however, always possible to remove the terms of first degree.

EXERCISE XXIV

Reduce, where possible, the following equations to a standard form of this chapter. Determine the axes, position of centers, vertices, and foci of ellipses and hyperbolas; asymptotes of hyperbolas; and foci, vertices, and directrices of parabolas. Sketch the curves.

1. $x^2 - 6x - 4y + 1 = 0$.
2. $9x^2 + 4y^2 - 36x - 24y + 36 = 0$.
3. $9x^2 - y^2 + 36x + 2y + 35 = 0$.
4. $16x^2 - y^2 - 80x - 6y + 75 = 0$.
5. $3x^2 + y^2 + 5x + 7y - 8 = 0$.
6. $5x^2 - 4y^2 + 10x - 16y = 0$.
7. $3x^2 + 4x - y + 7 = 0$.
8. $29x^2 + 16xy + 41y^2 - 45 = 0$.
9. $21x^2 + 52\sqrt{2}xy - 68y^2 - 324 = 0$.
10. $16x^2 - 24xy + 9y^2 - 180x + 10y - 75 = 0$.
11. $xy = 8$.
12. $xy - x^2 = 5$.
13. $3x^2 - 4xy + 6y^2 + 5x - 8y = 0$.
14. $14x^2 + 45xy - 14y^2 - 12x + 11y - 2 = 0$.
15. $xy + 2x + y + 1 = 0$.

16. Prove that the equation of a parabola with vertex at (h, k) and axis parallel to the x-axis is

$$(y-k)^2 = 2p(x-h).$$

What is the equation if the vertex is at (h, k) and the axis is parallel to the y-axis?

17. Prove that the equations of an ellipse and an hyperbola with center at (h, k) and axes parallel to the coördinate axes are, respectively,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

and

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

18. What are the equations of the asymptotes of the hyperbola in example 17?

19. Prove that $xy = ax + by + c$ is the equation of an equilateral hyperbola with asymptotes parallel to the coördinate axes, if $-c \neq ab$. By a translation of axes reduce the equation to the form $xy = k$.

20. What are the equations of the asymptotes of the hyperbola in example 19?

21. Prove that $y = \dfrac{ax+b}{cx+d}$ is the equation of an equilateral hyperbola, if $ad \neq bc$, and that the asymptotes are $x = -\dfrac{d}{c}$, $y = \dfrac{a}{c}\cdot$

22. In the equation of example 21 let $a = 1$, $b = 2$, $d = 3$, and plot the curves for the following values of c: $c = 2$, 1.6, 1.4, 1, .1. Show that these curves all pass through the same two points on the axes.

If c be allowed to approach the limiting value 0, what limiting form does the hyperbola approach? What limiting form if c approaches $\frac{3}{2}$?

23. An hyperbola has the lines $x = 2$ and $y = 4$ as asymptotes. It passes through the point (4, 2). Find its equation.

CHAPTER VIII

GRAPHS OF TRIGONOMETRIC, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS. GRAPHS IN POLAR COÖRDINATES

97. The sine curve. Consider the graph of the equation $y = \sin x$.

Let x be the radian measure of an angle. Let x and y be taken as rectangular coördinates of points in the plane, the abscissa of any such point being the number of radians in the angle, and the ordinate being the sine of that angle. The following properties of the locus follow from the properties of the sine of the angle.

1. The locus is not symmetric with respect to either axis, but is symmetric with respect to the origin, since

$$\sin(-x) = -\sin x.$$

2. The locus cuts the x-axis where $x = 0, \pi, 2\pi, \cdots; -\pi, -2\pi, \cdots$, *i.e.* where $x = k\pi$, k being any positive or negative integer, or zero. It crosses the y-axis only at the origin.

3. All real values of x make y real. All real values of y between and including -1 and 1 make x real; all other values of y make x imaginary.

4. No finite values of either variable makes the other infinite.

5. As x increases from 0 to $\frac{\pi}{2}$, y increases from 0 to 1,

As x increases from $\frac{\pi}{2}$ to π, y decreases from 1 to 0,

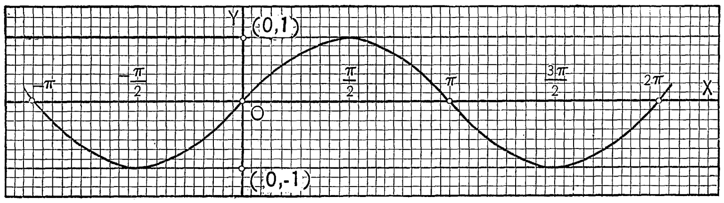
As x increases from π to $\frac{3\pi}{2}$, y decreases from 0 to -1,

As x increases from $\frac{3\pi}{2}$ to 2π, y increases from -1 to 0.

6. As x increases from 2π to 4π, y takes in succession the same set of values that it takes when x increases from 0 to 2π. In general, since $\sin(A+2k\pi)=\sin A$, where k is any positive or negative integer, it follows that if x is increased or decreased by any whole multiple of 2π, $\sin x$, or y, is unchanged. Hence if the curve be plotted through an interval of length 2π on the x-axis, other portions of the curve may be obtained from this portion by moving it to the right or left through the distance 2π, 4π, 6π, etc.

A few corresponding values are shown for x ranging from 0 to π, and the curve is drawn to pass through the points so determined. For x ranging from π to 2π the values of y are those given below changed in sign. (Fig. 93.)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π,
y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0.



$y = \sin x$

FIG. 93.

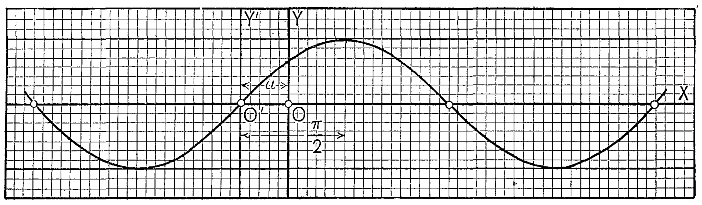
98. Periodic functions. A **periodic function** of a variable is a function whose value, for any value of the variable, is not changed by increasing the variable by a definite constant quantity. The least positive constant quantity by which the variable can be increased without changing the value of the function is called the **period** of the function.

Thus $\sin x$ is a periodic function of x, since $\sin(x+2\pi)=\sin x$. Since 2π is the least positive constant value by which x may be increased without changing the sine, 2π is the period of $\sin x$.

Again, $\tan\theta$ is a periodic function of θ, since $\tan(\theta+\pi)=\tan\theta$. The period is π.

Also $\cos(ax+b)$ is a periodic function of x with the period $\frac{2\pi}{a}$, since increasing x by $\frac{2\pi}{a}$ increases $ax+b$ by 2π, and this leaves the cosine unchanged.

99. Graph of $y=\sin(x+a)$. Let $x'=x+a$, or $x=x'-a$. This change of variable means geometrically a translation of axes to the new origin $(-a, 0)$. The equation referred to the new axes is $y'=\sin x'$. Figure 94 shows the curve and how it is located with respect to the axes.



$y=\sin(x+a)$

FIG. 94.

100. Graph of $y=\sin nx$, where n is positive. Let $x'=nx$, or $x=\frac{x'}{n}$. The equation then becomes $y=\sin x'$, the locus of which is shown in Fig. 93. Now the substitution of $x=\frac{x'}{n}$ can be interpreted as shortening the abscissas of all points in the ratio $1:n$ without changing the ordinates. If, then, the curve $y=\sin x$ be drawn, the curve $y=\sin nx$ can be obtained from it by shortening the abscissas of all points on the curve $y=\sin x$ in the ratio $1:n$. This is equivalent to compressing uniformly

in the direction of the x-axis any portion of the curve $y = \sin x$ which begins at the origin into $\frac{1}{n}$ of its original space, the end of the curve at the origin to remain at the origin.

It is also equivalent to choosing a unit on the x-axis equal to $\frac{1}{n}$ of the unit on the y-axis and then plotting the curve $y = \sin x$.

If n is less than 1, the contraction of the curve becomes in fact an expansion.

Graphs of $y = \sin 3x$ and of $y = \sin\left(\frac{2x}{3}\right)$ are shown in Fig. 95, together with $y = \sin x$.

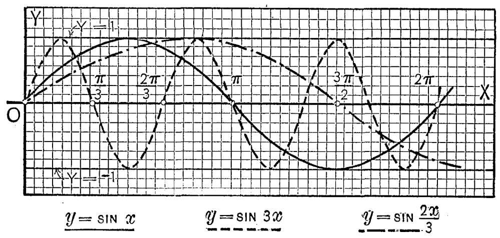


Fig. 95.

101. Graph of $y = \sin(nx + m)$. Letting $x = x' - \frac{m}{n}$, the equation becomes $y = \sin nx'$. Hence translate the axes to the new origin $\left(-\frac{m}{n}, 0\right)$ and construct the curve $y = \sin nx$. Compare Arts. 99 and 100.

Figure 96 shows the locus of $y = \sin(nx + m)$ for $n = 2$, $m = -1$.

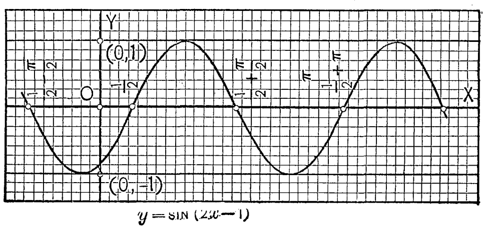
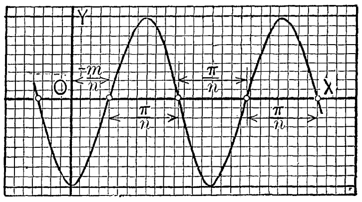


Fig. 96.

102. Graph of $y = p \sin(nx + m)$. The graph of this equation can be obtained by multiplying each ordinate of $y = \sin(nx + m)$ by p. The curve is shown in Fig. 97 for p and n both positive.



$y = p \sin(nx + m)$
FIG. 97.

103. The exponential curve, $y = a^x$, where a is positive.

(Only positive values of y are considered.)

(1) The locus is not symmetric with respect to either coördinate axis or the origin.

(2) It intersects the y-axis at (0, 1), but does not meet the x-axis.

(3) For every real value of x there is one real and positive value of y. Only this value is considered.

(4) No finite value of x makes y infinite.

(5) If $a < 1$, y approaches zero as x becomes infinite positively.
If $a > 1$, y approaches zero as x becomes infinite negatively.

(6) If $a > 1$, y increases always as x increases.
If $a < 1$, y decreases always as x increases.
If $a = 1$, the curve becomes the straight line $y = 1$.

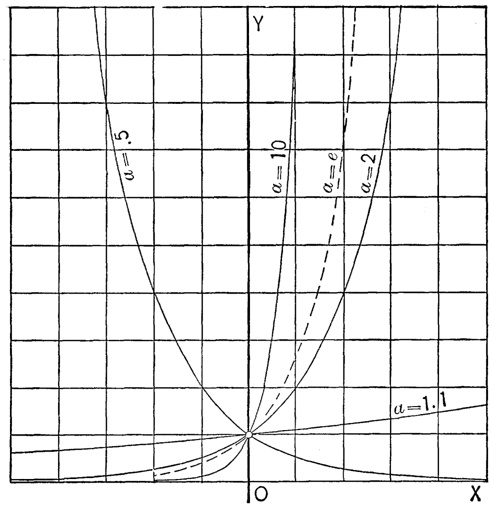
Figure 98 shows a few curves whose equations are of the form $y = a^x$, for certain values of a.

Values of y may be computed by logarithms.

E.g. if $a = e = 2.718 \cdots$*

* The quantity $1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 5} + \cdots = 2.71828 \cdots$ is denoted by e and is called the natural base of logarithms. It is of much importance in more advanced mathematical work.

then $$\log_{10} y = x \log_{10} 2.718 \cdots$$
$$= .4343\,x.$$



$y = a^x$.

FIG. 98.

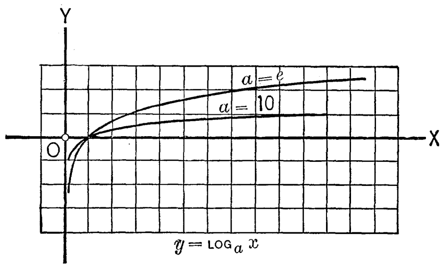
The following points are on the curve $y = e^x$,

x	-5	-3	-2	-1	0	1	2	3	5,
y	.007	.05	.14	.37	1	2.7	7.4	20	148.

After a few points on the curve have been obtained, other points are easily found by noticing that when x is doubled, y is squared; when x is tripled, y is cubed; etc. This follows at once from the law of exponents, $a^{nx} = (a^x)^n$.

104. The logarithmic curve, $y = \log_a x$. This curve is the same as that of $y = a^x$ with x and y interchanged. The curves for $y = \log_{10} x$ and $y = \log_e x$ are shown in Fig. 99.

Since $\log_b x = \log_a x \log_b a$, when the curve $y = \log_a x$ has been constructed for any value of a, the curve $y = \log_b x$ can be easily obtained from it by multiplying all the ordinates of the first curve by $\log_b a$.



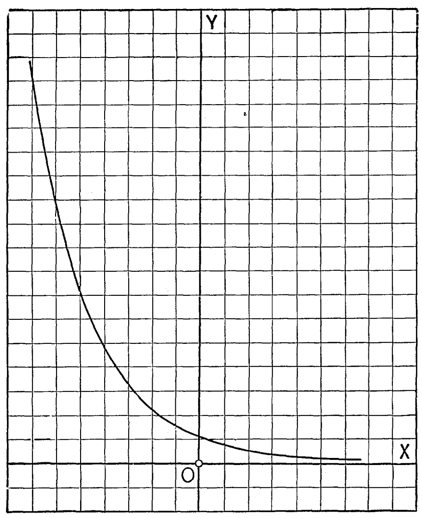
$y = \log_a x$

FIG. 99.

E.g. the ordinates of $y = \log_e x$ are 2.3026 times the corresponding ordinates of $y = \log_{10} x$, since

$$\log_e 10 = \frac{1}{\log_{10} e} = \frac{1}{.4343} = 2.3026.$$

105. Graph of $y = e^{-ax}$, where $e = 2.718 \cdots$ and a is positive.



$y = e^{-ax}$.

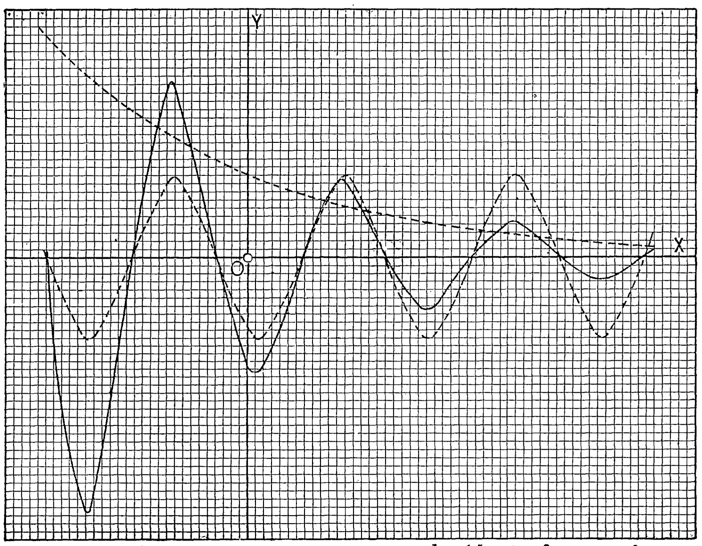
FIG. 100.

Since $e^{-ax} = (e^{-a})^x = \left(\frac{1}{e^a}\right)^x$, this curve is of the same form as $y = a_0^x$ where a_0 is less than 1. The curve is therefore as shown in Fig. 100. (Compare Art. 103.)

The rapidity with which the curve falls as the tracing point moves from left to right depends upon the value of a.

106. Graph of $y = be^{-ax} \sin (nx + m)$. This graph is easily obtained by plotting sep-

arately the graphs of $y = e^{-ax}$ and $y = b \sin(nx + m)$ and multiplying together the corresponding ordinates. The form of the curve is shown in Fig. 101. This is an important curve in the theory of alternating currents.



$y = b\, e^{-ax} \sin(nx + m)$, WHERE $a = -.4$, $b = 1.5$, $n = 2$, $m = -3$.

FIG. 101.

EXERCISE XXV

Plot the following curves. (The letters i, q, t, are variables.)

1. $y = \cos x$.
2. $y = \tan x$. (Divide ordinates of sine curve by those of cosine curve.)
3. $y = \csc x$. (Obtain from sine curve.)
4. $y = \sec x$.
5. $y = \dfrac{x}{\sin x}$. (Examine carefully near $x = 0$.)
6. $y = \cos 2x$.
7. $y = \cos 3x$.
8. $y = \tan 2x$.
9. $y = \tan 3x$.

10. $y = \sin(3x - 1)$.

11. $y = \cos(x + a)$.

12. $y = \cos(nx)$.

13. $y = \cos(nx + m)$.

14. Show that the graph of $y = \cos x$ is the same as the graph of $y = \sin x$ moved parallel to the x-axis the distance $\frac{\pi}{2}$ in the negative direction.

15. By what change in position can the graph of $y = \cot x$ be made to coincide with the graph of $y = \tan x$?

16. $i = be^{-at}$.

17. $i = b(1 - e^{-at})$. (Combine the graphs of $i = b$ and $i = be^{-at}$.)

18. $i = bte^{-at}$.

19. $q = b + c(1 + kt)e^{-at}$.

20. $q = a \sin nt + b \sin 3\,nt$.

21. $y = x + \sin x$.

22. $y = \sin x + \cos x$.

23. $y = \sin\left(\frac{1}{x}\right)$.

24. $y = \sin^2 x$.

26. $y = \sin^{11} x$.

25. $y = \sin^3 x$.

28. $y = \sin x + \sin 2\,x$.

27. $y^{11} = \sin x$.

30. $y^{10} = \sin x$.

29. $y = \sin^{10} x$.

32. $y = \sin x + \sin 3\,x + \sin 5\,x$.

31. $y = \sin^n x$.

34. $q = e^{-t} \sin 2\,t$.

33. $y = e^{-2x} \sin x$.

36. $y = \sin^{-1} x$.

35. $i = e^{-at} \sin nt$.

38. $y = \dfrac{\sin^{-1} x}{\cos^{-1} x}$.

37. $y = \tan^{-1} x$.

39. $y = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5\,x + \frac{1}{7} \sin 7\,x$.

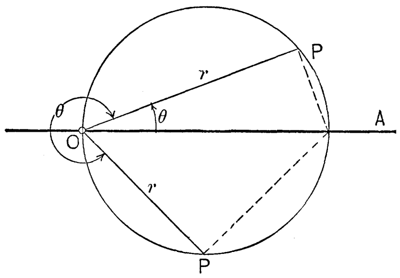
107. Plotting in polar coördinates. The methods used in plotting a curve in polar coördinates do not differ essentially from those used in plotting curves in rectangular coördinates. The difference comes mainly in the manner of locating the points. The following examples will sufficiently illustrate the methods.

Example 1. To plot in polar coördinates the curve whose equation is $r = a \cos \theta$.

The following pairs of values of r and θ may be at once written, using approximate values of r:

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
r	a	$.87\,a$	$.71\,a$	$.5\,a$	0	$-.5\,a$	$-.71\,a$	$-.87\,a$	$-a$
θ	210°	225°	240°	270°	300°	315°	330°	360°	
r	$-.87\,a$	$-.71\,a$	$-.5\,a$	0	$.5\,a$	$.71\,a$	$.87\,a$	a	

An examination of the variation in r as θ increases from 0 to 360° shows that as θ increases from 0 to 90°, r decreases from a to 0; as θ increases from 90° to 180°, r is negative and decreases from 0 to $-a$, the point (r, θ) tracing out a part of the curve in the fourth quadrant; as θ increases from 180° to 270°, r remains negative and increases from $-a$ to 0, the point (r, θ) tracing over again the part of the curve already traced in the first quadrant; as θ increases from 270° to 360°, r increases from 0 to a, the point (r, θ) tracing over again the part of the curve already traced in the fourth quadrant.



$r = a \cos \theta.$

FIG. 102.

If θ is allowed to increase beyond 360° or to take negative values, $\cos \theta$ takes on the same series of values already obtained, since $\cos(\theta \pm 360°) = \cos \theta$, and no new points are obtained. The curve is therefore as represented in Fig. 102.

The curve appears to be a circle. That it is so in fact may be proved as follows: Take any point $P(r, \theta)$ on the curve; then $r = a \cos \theta$ or $\cos \theta = \frac{r}{a}$. Therefore $\angle\ OPA$ must be a right angle. Therefore the curve is a circle.

EXAMPLE 2. To plot in polar coördinates the curve whose equation is $r^2 = a^2 \cos 3\theta$.

For every value of θ which makes $\cos 3\theta$ positive there are two values of r which differ only in sign, and for every value of θ which makes $\cos 3\theta$ negative the values of r are imaginary. In order then for r to be real, 3θ must be an angle in the first or fourth quadrant.

Let the positive value of r be taken for discussion first.

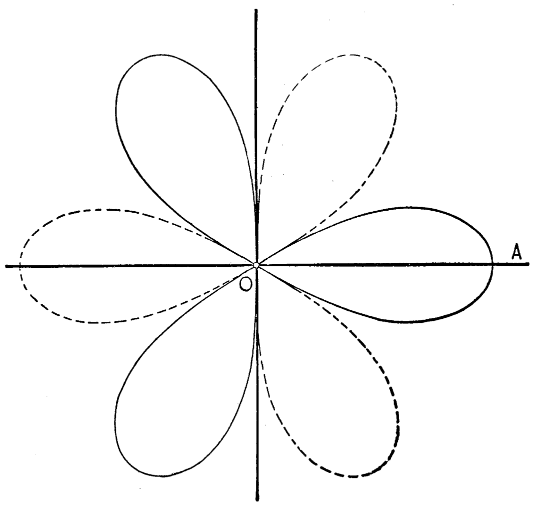
The following table shows the changes that take place in r as θ increases from 0 to 2π.

θ	0 to $\frac{\pi}{6}$	$\frac{\pi}{6}$ to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to $\frac{2\pi}{3}$	$\frac{2\pi}{3}$ to $\frac{5\pi}{6}$	$\frac{5\pi}{6}$ to $\frac{7\pi}{6}$
3θ	0 to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to 2π	2π to $\frac{5\pi}{2}$	$\frac{5\pi}{2}$ to $\frac{7\pi}{2}$
r	a to 0	imag.	0 to a	0 to a	imag.

θ	$\frac{7\pi}{6}$ to $\frac{4\pi}{3}$	$\frac{4\pi}{3}$ to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to $\frac{11\pi}{6}$	$\frac{11\pi}{6}$ to 2π
3θ	$\frac{7\pi}{2}$ to 4π	4π to $\frac{9\pi}{2}$	$\frac{9\pi}{2}$ to $\frac{11\pi}{2}$	$\frac{11\pi}{2}$ to 6π
r	0 to a	a to 0	imag.	0 to a

The second column is to be read, as θ increases from 0 to $\frac{\pi}{6}$, 3θ increases from 0 to $\frac{\pi}{2}$, and hence r decreases from a to 0, and similarly for the other columns.

A few intermediate values of r and θ, computed from a table of natural cosines, are shown for values of θ ranging from 0 to $\frac{\pi}{6}$.



$r^2 = a^2 \cos 3\theta$

FIG. 103.

θ	5°	10°	15°	20°	25°	30°
r	$.98\,a$	$.93\,a$	$.84\,a$	$.71\,a$	$.51\,a$	0

Since $\cos 3\theta$ takes the same values, either in the same order or in the reverse order, when θ increases through the other intervals for which r is real as it does when θ increases from 0 to $\frac{\pi}{6}$, the values of r are the same in those intervals as in the first.

The curve is shown in Fig. 103. The dotted portion is the part corresponding to the negative values of r.

If θ is allowed to increase beyond 2π, or to take negative values, $\cos 3\theta$ takes the same set of values over again, and the same points of the curve are again obtained.

EXERCISE XXVI

Plot the following curves in polar coördinates.

1. $r = a \sin \theta$. (Prove it is a circle.) 2. $r = \theta$.
3. $r = a \tan \theta$. 4. $r = 2\theta$. 5. $r = a \cos 2\theta$.
6. $r \cos \theta = a$. (Prove it is a straight line.)
7. $r \sin \theta = a$. (Prove it is a straight line.)
8. $r\theta = C$. (Called hyperbolic spiral.)
9. $r = a\theta$. (Called logarithmic spiral.)
10. $r = a(1 - \cos\theta)$. (The cardioid.)
11. $r = a(1 + \cos\theta)$. (The cardioid.)
12. $r = \dfrac{a}{(1 - \cos\theta)}$. (Prove it is a parabola by transforming to rectangular coördinates.)
13. $r = \dfrac{4}{1 - .5 \cos\theta}$. (Prove it is an ellipse.)
14. $r = \dfrac{4}{1 - 2\cos\theta}$. (Prove it is an hyperbola.)
15. $r = a \sin 2\theta$. 16. $r = a \sin 3\theta$. 17. $r = a \cos 3\theta$.
18. $r = a \cos 4\theta$. 19. $r^2 = a^2 \cos 2\theta$. 20. $r^2 = a^2 \sin 2\theta$.
21. $r^2 = a^2 \sin 3\theta$. 22. $r^2 = a^2 \sin\left(\dfrac{\theta}{2}\right)$. 23. $r = 8 \cos\left(\dfrac{\theta}{3}\right)$.
24. $r = a \sin\left(\dfrac{3\theta}{2}\right)$. 25. $r = 1 - 2\cos\theta$. 26. $r = 2 - \cos\theta$.
27. $r = 2a\cos\theta + b$, where b takes the values 0, a, $2a$, $3a$.

CHAPTER IX

PARAMETRIC EQUATIONS OF LOCI

108. Parametric equations. A single equation connecting two variables, which can be solved for one of the variables, may always be replaced by two equations which express the value of each of the variables of the original equation in terms of a third variable. Moreover, one of the two equations may have any form whatever.

Thus in the equation of the circle, $x^2 + y^2 = r^2$, a third variable, t, may be introduced by letting x be equal to some function of t; substituting this value of x in $x^2 + y^2 = r^2$ the value of y may be found in terms of t. *E.g.* if $x = r \cos t$, then $y = \pm r \sin t$.

It often happens that it is easier to obtain the values of coördinates of points on a given locus in terms of some third variable than it is to obtain an equation directly connecting the coördinates of the points, and in some cases the two equations can be obtained where it is not possible to obtain the equation directly connecting the coördinates of the points.

The third variable in terms of which the coördinates of the points are expressed is called the **parameter**, and the two equations are called the **parametric equations** of the locus.

Frequently the parameter may be given an interesting geometric interpretation.

109. The parametric equations of the circle. Let the center of the circle be at the origin and let the radius be r. Let θ be the angle which the radius to the point (x, y) on the circle makes with the x-axis. Then

$$x = r \cos \theta, \; y = r \sin \theta.$$

These equations hold for every point on the circle and hence represent the circle completely. They are parametric equations of the circle.

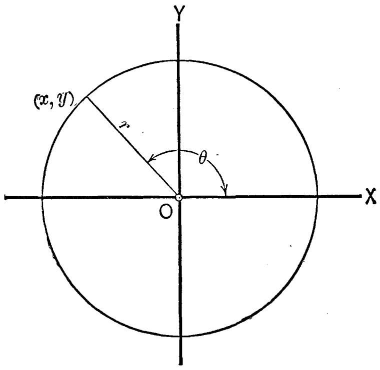


FIG. 104.

If θ be eliminated from the equations (by squaring and adding), the ordinary equation, $x^2 + y^2 = r^2$, is obtained.

110. The parametric equations of the ellipse. Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Draw a circle with center at origin and radius a. Through any point $P(x, y)$ of the ellipse draw a line parallel to the y-axis to meet the circle in P' on the same side of the x-axis as P. Draw OP' and let the inclination of OP' be θ. Then $x = a \cos \theta$. Substituting $a \cos \theta$ for x in the equation of the ellipse, there results $y = \pm b \sin \theta$. Since it was agreed to take P' and P on the same side of the x-axis, the plus sign must be taken in the value of y. Hence

$$\boldsymbol{x = a \cos \theta, \quad y = b \sin \theta,}$$

are the parametric equations of the ellipse.

The angle θ is called the **eccentric angle.**

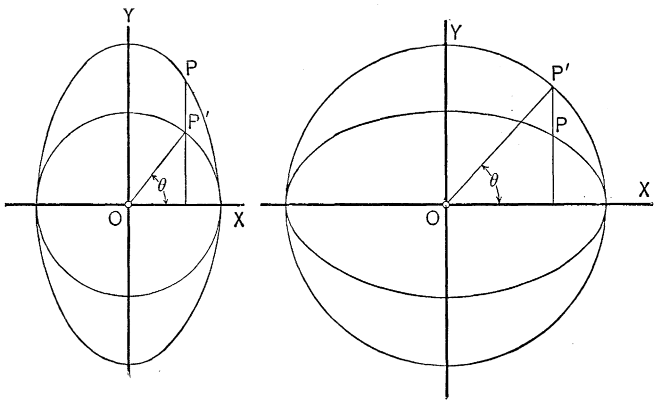


FIG. 105.

111. Construction of the ellipse. To construct an ellipse of semi-axes a and b, $a > b$, take the center of the ellipse as a center and describe circles of radii a and b. Draw any radius making an angle θ with the major axis. Through the points where the radius cuts the inner and outer circles draw parallels respectively to the minor and major axes. Their intersection is a point of the ellipse.

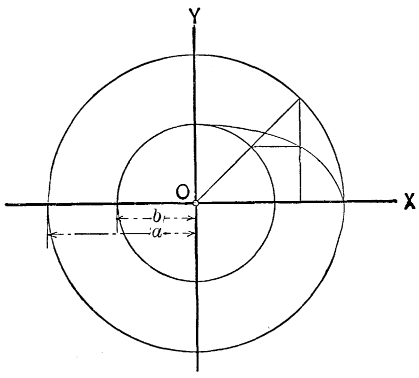


FIG. 106.

PROOF. Taking the x-axis along the major axis of the ellipse the point of intersection P is at once seen to have the coördinates $x = a \cos \theta$, $y = b \sin \theta$, and is therefore a point of the ellipse from the preceding article.

EXERCISE 1. Construct an ellipse by this method.

Exercise 2. Prove that, for the same values of x, the ordinates of the ellipse and circle in Fig. 105 have a constant ratio, $\frac{b}{a}$.

Exercise 3. The sun's rays fall vertically upon a plane; prove that the shadow on this plane of a circular hoop not parallel to the plane is an ellipse.

112. The cycloid. The curve traced by a fixed point on the circumference of a circle as the circle rolls in a plane along a fixed straight line is called the **cycloid.**

The circle is called the **generator circle** and the point the **generating point.**

To derive the equations of the cycloid: Let the fixed line be taken as x-axis and the point on this line where the generating point touches it as the origin. Take the y-axis perpendicular to the x-axis.

Let $P(x, y)$ be any position of the generating point, θ the angle, measured in radians, through which the radius through P has turned since the generating point left the origin, and a the radius of the circle. Then (Fig. 107)

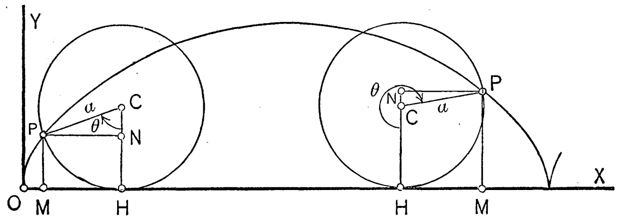


Fig. 107.

$$x = OM = OH + HM.$$

$$y = MP = HC + CN.$$

Now $OH = \text{arc } HP = a\theta,$

$$HM = -PN = -a \sin \theta,$$

$$HC = a,$$
$$CN = -a \cos\theta.$$
$$\therefore \left.\begin{aligned} \boldsymbol{x} &= \boldsymbol{a\theta - a \sin\theta}, \\ \boldsymbol{y} &= \boldsymbol{a - a \cos\theta}. \end{aligned}\right\} \tag{1}$$

(The student should make sure that these equations hold for either position of the generator circle shown in Fig. 107, and should draw other positions of the generator circle and prove that the same equations hold.)

Equations (1) give the values of x and y in terms of a third variable θ. By assigning values to θ, values of x and y may be computed and thus points on the curve located.

It is usual to take the two equations (1) as representing the cycloid, but a single equation connecting x and y may be obtained as follows:

From the second equation, $1 - \cos\theta = \dfrac{y}{a}$, or $\operatorname{vers}\theta = \dfrac{y}{a}$.

$$\therefore \theta = \operatorname{vers}^{-1}\frac{y}{a},$$

and $$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{a-y}{a}\right)^2} = \frac{1}{a}\sqrt{2ay - y^2}.$$

Substituting these values of θ and $\sin\theta$ in the first of eqs. (1), there results

$$\boldsymbol{x = a \operatorname{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}}. \tag{2}$$

113. Construction of the cycloid. Besides the method of locating points on the cycloid by computing values of x and y from eqs. (1) of Art. 112, the following method may be easily employed: On a straight line lay off a distance OA equal to the circumference of the generating circle. At the middle point B of OA draw a circle equal to the generating circle tangent to OA. Divide OB into a number of equal parts by the points B_1, B_2, B_3, etc., and the semi-circumference BC into the same number of equal parts by the points C_1, C_2, C_3, etc.,

obtained by use of the protractor. Through C_1, C_2, C_3, $\cdots$ draw lines parallel to OA.

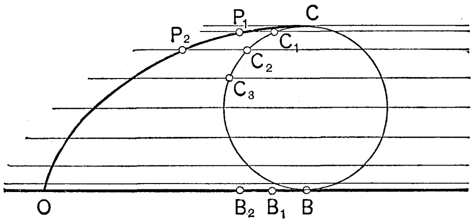


FIG. 108.

As the circle rolls back, the point P, now at the top of the circle, generates the cycloid, the point P descending to the level of C_1 when the point of tangency moves back to B_1. Hence the point P_1 may be obtained by using C_1 as a center and BB_1 as a radius to describe an arc cutting the line through C_1.

Similarly with radius equal to BB_2 and center C_2 the point P_2 is obtained, etc.

Other methods of constructing the cycloid are employed by draftsmen.

EXERCISE 1. Construct a cycloid by the method explained, dividing the circumference into twelve equal parts.

EXERCISE 2. Construct a cycloid by computing values of x and y by eqs. (1), Art. 112.

114. The hypocycloid. The hypocycloid is the curve traced by a fixed point on the circumference of a circle which rolls internally along the circumference of a fixed circle.

To derive the equations of the hypocycloid: Let the radii of the fixed and rolling circles be a and b respectively. Take the center of the fixed circle as origin, and the line through this center and the point of contact of the generating point with the fixed circle as x-axis. Let $P(x, y)$ be any position of the generating point, θ the angle through which the line of centers has rotated, and ϕ the angle through which any

radius of the generator circle has turned since the generating point left the x-axis. Then (Fig. 109),

$$x = OM = OH + NP = OC \cos\theta + CP \cos\phi$$
$$= (a-b)\cos\theta + b\cos\phi,$$
$$y = MP = HC - NC = (a-b)\sin\theta - b\sin\phi.$$

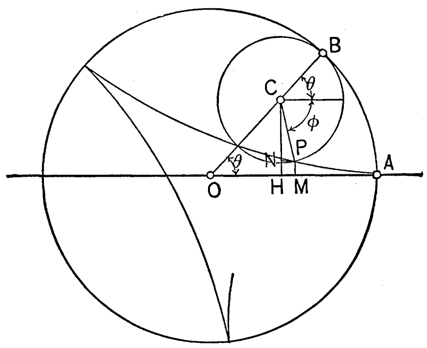


Fig. 109.

Now arc PB = arc AB, and therefore $b(\phi + \theta) = a\theta$,

or $$\phi = \frac{a-b}{b}\theta.$$

$$\therefore\ \boldsymbol{x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)},$$

$$\boldsymbol{y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)}.$$

115. Construction of the hypocycloid. From the above equations as many values of x and y as desired may be computed by assigning arbitrary values to θ. By this means a sufficient number of points may be obtained, through which the curve may be drawn.

Another method is as follows: Draw two concentric circles, K and K', with radii a and $a-b$ respectively. Let $\phi' = \theta + \phi$;

then $a\theta = b\phi'$. Compute the value of θ which makes $\phi' = 360°$, *i.e.* $\theta = \frac{b}{a}\,360°$. Let AOB be this angle, constructed by use of the protractor. Then B is the second point of contact of the generating point with the fixed circle. Divide AOB into any number, n, of equal parts and draw radii to intersect the circle K' at C_1, C_2, C_3, etc., and the circle K at B_1, B_2, B_3, etc. With C_1, C_2, $\cdots$ as centers draw circles of radius b.

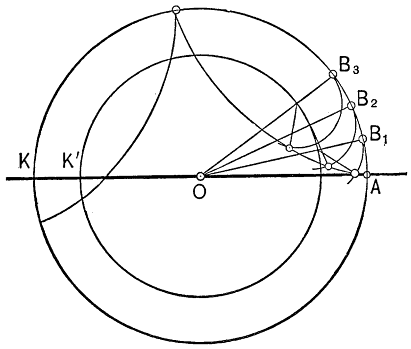


FIG. 110.

The position of the generating point on the first of these circles is obtained by drawing an angle $B_1C_1P_1$ equal to $\frac{1}{n}$th of 360°; the point on the second circle by drawing an angle $B_2C_2P_2$ equal to $\frac{2}{n}$ths of 360°, etc. See Fig. 110, where $n = 8$.

116. The hypocycloid where $a = 2\,b$. Letting $a = 2\,b$ in the equations of Art. 114, there is obtained

$$x = a \cos\theta,$$
$$y = 0.$$

The latter equation shows that the generating point moves along the x-axis, and the former that it is at any time in the same vertical as the point of contact of the circles.

Hence, *if a circle rolls within a fixed circle of double the diameter, every point of the rolling circle moves back and forth along a diameter of the fixed circle.* Moreover, if the circle rolls with uniform angular velocity, every point of it moves with simple harmonic motion.*

117. The four-cusped hypocycloid. The points where the generating point reverses its direction of motion are called cusps. Thus the points of contact of the generating point and the fixed circle are cusps.

If $a = 4b$ there are four cusps. The curve in this case is of interest because it is possible to eliminate θ between the equations of Art. 114 and obtain a simple equation connecting x and y.

Substituting $\frac{a}{4}$ for b in eqs. (1), Art. 114, they become

$$x = \frac{3a}{4}\cos\theta + \frac{a}{4}\cos 3\theta = \frac{a}{4}(3\cos\theta + \cos 3\theta),$$
$$y = \frac{3a}{4}\sin\theta - \frac{a}{4}\sin 3\theta = \frac{a}{4}(3\sin\theta - \sin 3\theta).$$

By trigonometry,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Substituting these values, there result

$$x = a\cos^3\theta,$$
$$y = a\sin^3\theta,$$

from which

$$\cos\theta = \left(\frac{x}{a}\right)^{\frac{1}{3}},$$
$$\sin\theta = \left(\frac{y}{a}\right)^{\frac{1}{3}}.$$

* When a point moves with uniform velocity along the circumference of a circle the projection of the point on any diameter is said to have simple harmonic motion.

Squaring, adding, and clearing of fractions,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

118. The epicycloid. The **epicycloid** is the curve traced by a fixed point on a circle which rolls externally on the circumference of a fixed circle.

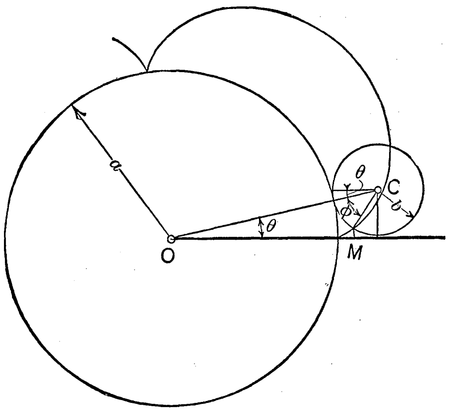


Fig. 111.

Let the student show from the figure that the equations are

$$x = (a + b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a + b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

Notice that the equations differ from those of the hypocycloid only in having $-b$ take the place of b.

119. The cardioid. The epicycloid for which the rolling and fixed circles are equal is called the **cardioid.** Its equations are obtained by letting $b = a$ in the equations of the

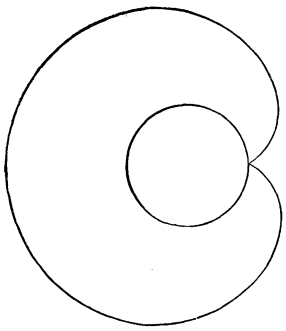


Fig. 112.

preceding article. They then become

$$\boldsymbol{x = 2\,a \cos\theta - a \cos 2\,\theta,}$$
$$\boldsymbol{y = 2\,a \sin\theta - a \sin 2\,\theta.}$$

120. The involute of the circle. If a thread is wound around a circular form and then unwound, kept always stretched, any point of the thread traces a curve called the **involute of the circle.**

To derive its equations: Choose the axes as in Fig. 113. Let a be the radius of the circle, $P(x, y)$ the position of the generating point at any time, and θ the angle through which the radius to the point of tangency has turned during the unwinding. Then

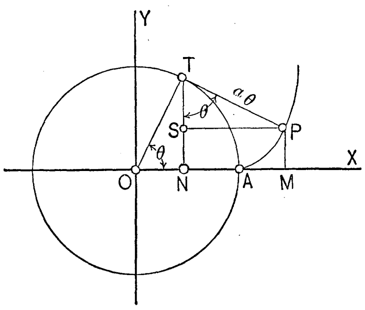


FIG. 113.

$$x = OM = ON + NM = ON + TP \sin\theta,$$
$$y = MP = NT - ST = NT - TP \cos\theta.$$

Now $\quad TP = \text{arc } AT = a\theta.$

$$\therefore\ \boldsymbol{x = a \cos\theta + a\,\theta \sin\theta,}$$
$$\boldsymbol{y = a \sin\theta - a\,\theta \cos\theta,}$$

are the equations of the involute of the circle.

EXERCISE XXVII

1. Prove that if a circle of radius a rolls along a straight line, a point on a fixed radius of the circle at a distance b from the center describes a curve whose equations are

$$x = a\theta - b \sin\theta,\ \ y = a - b\cos\theta.$$

Plot the curve for $b < a$; for $b > a$.

These curves are called **trochoids.**

2. Devise a method of constructing the cycloid similar to the method of constructing the hypocycloid in Art. 115.

3. Carefully construct on coördinate paper a cycloid by the method you have described.

By counting the squares between the cycloid and the line on which the circle rolls and the squares in the generating circle, what idea do you get of the area of the cycloid?

4. By combining the equations of the cardioid (Art. 119) and transforming to polar coördinates, show that the polar equation of the cardioid is $\boldsymbol{r = 2a(1 - \cos\theta)}$, where the pole is the point of contact of the generating point with the fixed circle.

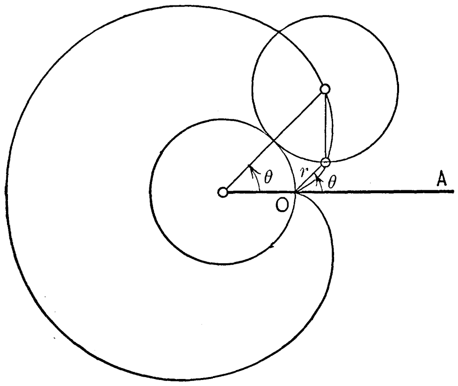


Fig. 114.

Suggestion. Square and add the equations of Art. 119, move to new origin by letting $x = x' + a$, $y = y'$; substitute $x' = r\cos\theta$, $y' = r\sin\theta$; complete the square in the terms in r, and extract the square root. Also derive the polar equation independently from the figure. (Fig. 114.)

5. Taking the origin at the point of the cycloid farthest from the line on which the circle rolls, and the x-axis parallel to that line show that the equations of the cycloid are

$$x = a\theta + a\sin\theta,\ y = -a + a\cos\theta,$$

where θ is measured from the positive direction of the y-axis to the radius of the circle through (x, y), clockwise rotation being counted positive.

6. Construct a hypocycloid where $a = 3b$.

7. Devise a method for constructing the epicycloid and apply it to the case where $a = 4b$.

8. Construct the involute of a circle.

9. A circle rolls along a straight line, and a line through the center of the circle turns about a point of the fixed line. Find the equations of the locus of the point of intersection of line and circle, and plot the curve.

Ans. $x = a(\cot\theta + \cos\theta)$,

$y = a(1 + \sin\theta)$ for outer point,

and $x = a(\cot\theta - \cos\theta)$,

$y = a(1 - \sin\theta)$ for the inner point.

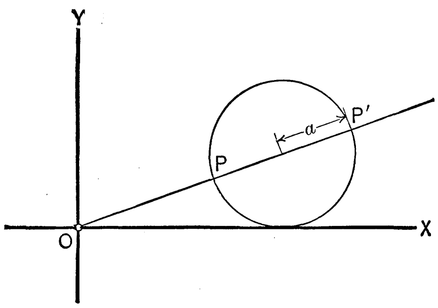


FIG. 115.

10. Show that the polar equations of the curves of example 9 are $r = a(\csc\theta + 1)$ and $r = a(\csc\theta - 1)$ respectively.

11. A circle moves with its center always on a straight line, and a second straight line passes through the center of the circle and a fixed point. Find the loci of the points of intersection of the second line and the circle.

Ans. Using the notation of Fig. 116,

$x = b\tan\theta + a\sin\theta$,

$y = a\cos\theta$, for P.

$x = b\tan\theta - a\sin\theta$,

$y = -a\cos\theta$, for P'.

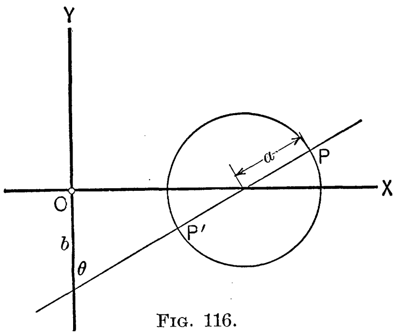


FIG. 116.

12. Plot the curves of example 11 for $b < a$, $b = a$, $b > a$.

CHAPTER X

INTERSECTIONS OF CURVES. SLOPE EQUATIONS OF TANGENTS

121. Intersections of curves. It has been seen that an equation in two variables can be represented graphically by a curve, every point of which has coördinates which satisfy the equation. Two different equations in the same two variables will then in general represent two different curves. If these curves be plotted on the same diagram they may or may not intersect. The coördinates of the points of intersection, if any, must satisfy both equations, and no other points will have this property. Now the values of the variables which satisfy two equations are obtained by solving the two equations as simultaneous. Hence to find the points of intersection of two curves, solve the equations of the curves as simultaneous. The real values of the variables so obtained which satisfy both equations are the coördinates of the points of intersection of the curves.

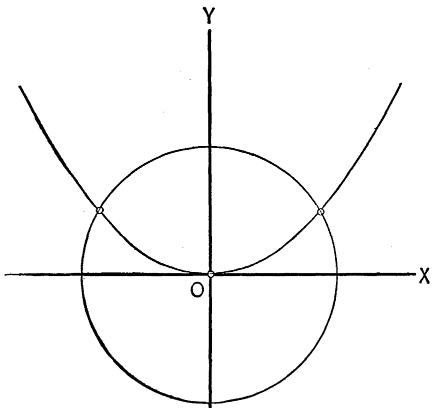


Fig. 117.

Example. To find the points of intersection of the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$. Eliminating x from the first equation by substitut-

ing the value of x from the second, there results

$$y^2 + 6y - 16 = 0,$$

from which $y = 2$ or -8. Substituting the first of these values of y in the second equation, there is obtained $x = \pm 2\sqrt{3}$. The substitution of -8 in the second equation gives imaginary values of x. Hence the points of intersection are $(2\sqrt{3}, 2)$ and $(-2\sqrt{3}, 2)$, or approximately $(3.46, 2)$ and $(-3.46, 2)$.

On plotting the curves these results are seen to be approximately correct.

EXERCISE XXVIII

Find the points of intersection of the following pairs of curves. Check graphically by plotting the curves and measuring the coördinates of the points of intersection.

1. $x^2 + y^2 = 5, \ y^2 = 4x$.
2. $y = 3x + 7, \ x^2 + y^2 = 9$.
3. (*a*) $y = 2x + \frac{1}{2}, \quad y^2 = 4x$. *Ans.* $(\frac{1}{4}, 1)$ (Tangent).
   (*b*) $y = 2x + .49, \ y^2 = 4x$. *Ans.* $(.326, 1.141), (.184, .859)$.
   (*c*) $y = 2x + .51, \ y^2 = 4x$. *Ans.* No intersection.
4. $x^2 + 4y^2 = 16, \ x^2 + y = 0$.
5. $3x - y = 1, \ 16x^2 + 9y^2 = 144$.
6. $x + y = 5, \ 9x^2 + 16y^2 = 144$.
7. $x^2 + y^2 = 16, \ x^2 - y^2 = 9$.
8. For what values of b is $y = 2x + b$ tangent to $x^2 + y^2 = 9$?
9. For what values of b is $y = mx + b$ tangent to $x^2 + y^2 = r^2$?
10. For what value of p is $y^2 = 2px$ tangent to $y = 3x + 1$?
11. Prove that the two segments of any line which cuts $xy = C$ in two points, included between the curve and its asymptotes, are equal.

122. Graphical solution of simultaneous equations. It frequently happens that when two equations containing two variables are given it is not possible to eliminate one of the variables, and so obtain an equation with only one variable; or, if the elimination is possible, the resulting equation may be very difficult or impossible of solution by ordinary methods.

In such cases, if the coefficients are numerical, an approximate solution may be obtained by carefully plotting the curves and measuring the coördinates of the points of intersection. More accurate solutions may then be obtained by methods illustrated in the following examples.

Example 1. To find the intersection of the curves

$$y = \sin x \tag{1}$$

and

$$y = 2x + 1. \tag{2}$$

Plot the curves carefully on coördinate paper.

From the figure the abscissa of the point of intersection is seen to be about $-.9$. Substitute this value in equations (1) and (2), remembering that .9 radian $= .9$ of $57°.3 = 51°34'$, and there results,

from (1) $\qquad y = \sin(-51°34') = -.78;$

from (2) $\qquad y = -.8.$

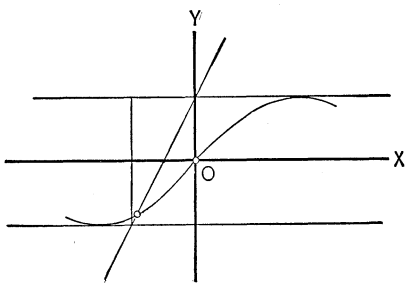


Fig. 118.

This shows the assumed value of x to be too small, but very near to the correct value. (Compare Fig. 118.)

Try next $x = -.88$.

Then, from (1), $\qquad y = \sin(-50°25') = -.771,$

from (2), $\qquad y = -.76.$

This shows the assumed value of x to be too large, so next try $x = -.89$.

Then, from (1), $y = \sin(-51°) = -.777$,

from (2), $y = -.78$.

Hence, correct to two significant figures, the solution is

$$x = -.89, \quad y = -.78.$$

EXAMPLE 2. To solve the equation

$$x^3 - 2x^2 + 4x - 7 = 0. \tag{1}$$

Let $$y = x^3 - 2x^2 + 4x - 7. \tag{2}$$

Then the solution of (1) is the same as the simultaneous solutions of (2) and the equation

$$y = 0. \tag{3}$$

Plot the curve of eq. (2). (Figure not shown.)

The following are corresponding values of x and y:

x	0	1	2	3	4	-1	-2	-3
y	-7	-4	1	14	41	-14	-31	-64

The curve is seen to cross the x-axis between 1 and 2, at about 1.8.

Try this value of x in eq. (2);

$$y = 5.832 - 6.48 + 7.2 - 7 = -.448.$$

Hence the value of 1.8 for x is too small.

Try next $x = 1.9$; then $y = .239$.

Hence the value of 1.9 for x is too large.

Plot now on an enlarged scale the points representing x and y for $x = 1.8$ and 1.9, and join the points by a straight line. Since the interval is small, the curve probably differs but slightly from a straight line in the interval. The line is seen to cross at about .65 of the distance from 1.8 to 1.9. Then 1.865 is probably a close approximation to a root of eq. (1). Substituting this value of x in eq. (2), there results $y = .0094$.

The work of computation, arranged according to Horner's method of synthetic division, is as follows:

1	-2	4	-7	)1.865
	1.865	$-.2518$	6.9906	
	$-.135$	-3.7482	$-.0094$	

By the *Remainder Theorem* from Algebra, the value of y is the last remainder, $-.0094$.

Since y comes out negative, it shows that in this case the assumed value of x is too small. Try then $x = 1.866$.

1	-2	4	-7	)1.866
	1.866	$-.2500$	6.9975	
	$-.134$	3.7500	$-.0025$	

Hence $y = -.0025$.

Try next $x = 1.867$:

1	-2	4	-7	)1.867
	1.867	$-.2483$	7.0044	
	.133	3.7517	.0044	

Hence $y = .0044$.

The root therefore lies between 1.866 and 1.867 and is nearer to the former. Hence, correct to four significant figures, a root of eq. (1) is 1.866.

Evidently one could by this method obtain a root correct to any desired degree of accuracy.

Example 3. To solve the equation

$$\phi^2 - \sin 2\phi = 0. \qquad (1)$$

This may be treated as in the last example, or it may be more easily solved as follows: Plot separately the curves

$$y = \phi^2 \qquad (2)$$

and

$$y = \sin 2\phi \qquad (3)$$

on the same diagram. Then a value of ϕ at a point of intersection of the curves of eqs. (2) and (3) is a root of eq. (1).

The figure shows that a value of ϕ at the intersection is a little less than 1. Try then $\phi = .9$.

Then from (2) $y = \ \ .81$
and from (3) $\underline{y = \ \ .974}$
Difference $= -.164.$

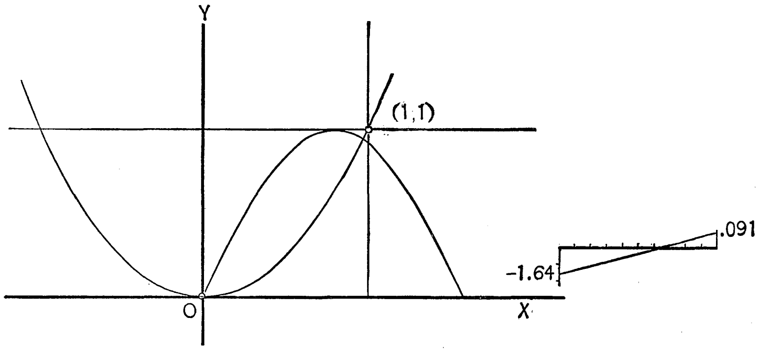


FIG. 119.

Substitute $\phi = 1,$
then from (2) $y = 1$
and from (3) $\underline{y = \ \ .909}$
Difference $= \ \ .091$

Plot on an enlarged scale the difference for $\phi = .9$ and $\phi = 1$, using ϕ as abscissa and difference as ordinate, and connect the points obtained by a straight line. This straight line is seen to cross the axis at about .65 of the distance from .9 to 1. On substituting $\phi = .965$ there results

from (2) $y = .931$
and from (3) $\underline{y = .936}$
Difference $= - \ .005$

Let the student show that when $\phi = .966$ and .967 the differences computed as above are $-$ .0022 and .0004, respectively,

and that hence the solution of eq. (1), correct to three significant figures, is $\phi = .967$.

EXERCISE XXIX

Solve the following pairs of equations :

1. $y = \cos x, \quad y^2 = 4x$.
2. $10y = x, \quad y = \log_{10} x$.
3. $s = \sin 3t, \quad s = \tan 2t$.
4. $x = \theta - \sin\theta, \quad x = 1 - \cos\theta$.
5. $y = x^2, \ y = 2^x$.

Solve the following equations by graphical methods :

6. $x^3 + 5 = 0$.
7. $x^3 - x + 7 = 0$.
8. $2\theta - \cos 2\theta = 0$.
9. $1 - \theta - \tan\theta = 0$.
10. $2^x - x + 1 = 0$.

123. Slope equations of tangents. Tangent to the ellipse. Let a line of slope m be drawn tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

To derive its equation.

Any line of slope m has an equation of the form

$$y = mx + k. \tag{2}$$

If eqs. (1) and (2) be solved as simultaneous, the points of intersection of the loci will be obtained. These intersections may be real and distinct, real and coincident, or imaginary, depending upon the value of k. It is evident from the figure that there are two values of k for which the line is tangent to the ellipse.

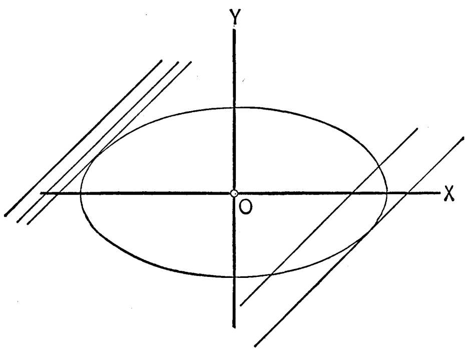


FIG. 120.

Substituting the value of y from eq. (2) in eq. (1) and collecting terms, there results

$$(b^2 + a^2m^2)x^2 + 2a^2mkx + a^2(k^2 - b^2) = 0. \qquad (3)$$

The roots of eq. (3) are the abscissas of the points of intersection of the line and the ellipse. In order that the line be tangent to the ellipse these values of x must be equal; and conversely, if they are equal, so also are the values of y obtained by substituting these values of x in eq. (2), and hence the line is a tangent. Now the condition that the roots of the equation $ax^2 + bx + c = 0$ be equal is $b^2 = 4ac$. Hence, the roots of eq. (3) are equal if

$$4a^4m^2k^2 = 4a^2(k^2 - b^2)(b^2 + a^2m^2),$$

which reduces to

$$k^2 = b^2 + a^2m^2.$$

Therefore the equations of the tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with slope m are

$$\boldsymbol{y = mx \pm \sqrt{a^2m^2 + b^2}}.$$

These equations are called the **slope equations** of the tangents to the ellipse.

124. Tangent equations for reference. The student should derive the following equations of tangents to the given curves.

CURVE	TANGENT
(1) $\boldsymbol{x^2 + y^2 = r^2}$,	$\boldsymbol{y = mx \pm r\sqrt{m^2 + 1}}$.
(2) $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$,	$\boldsymbol{y = mx \pm \sqrt{a^2m^2 + b^2}}$.
(3) $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,	$\boldsymbol{y = mx \pm \sqrt{a^2m^2 - b^2}}$.
(4) $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1}$,	$\boldsymbol{y = mx \pm \sqrt{b^2 - a^2m^2}}$.

(5) $\boldsymbol{y^2 = 2px,}$ $\qquad\qquad \boldsymbol{y = mx + \frac{p}{2m}.}$

(6) $\boldsymbol{x^2 = 2py,}$ $\qquad\qquad \boldsymbol{y = mx - \frac{pm^2}{2}.}$

(7) $\boldsymbol{xy = c,}$ $\qquad\qquad \boldsymbol{y = mx \pm 2\sqrt{-cm}.}$

EXERCISE XXX

(Use the above formulas in solving these exercises.)

1. Find the equations of tangents to $x^2 + y^2 = 16$ which have a slope equal to $\sqrt{3}$. Check graphically.

2. Find the equations of tangents to $9x^2 + 16y^2 = 576$ which are parallel to $y = x$. Check graphically.

3. Find the equation of a tangent to $y^2 = 6x$ which is perpendicular to $2x - y - 3 = 0$. Plot the lines. Where do they intersect?

4. Write the equation of a tangent to $y^2 = 2px$ and the equation of a line through the focus perpendicular to the tangent, and prove that they intersect on the y-axis.

5. Obtain the slope equation of a tangent to the circle from the equation of the tangent to the ellipse.

6. Find the equations of tangents to $y^2 = 6x$ from the exterior point $(2, 4)$. Check graphically.

7. Find the equations of tangents from $(7, 1)$ to $x^2 + y^2 = 25$. Check graphically. *Ans.* $3x + 4y - 25 = 0$, $4x - 3y - 25 = 0$.

8. Find the equations of tangents to $9x^2 - 25y^2 = 225$ which pass through $(-1, 3)$. Check graphically.
Ans. $x - y + 4 = 0$, $3x + 4y - 9 = 0$.

9. Find the equations of tangents to $12x^2 + 5y^2 = 30$ which intersect in $(-3, -2)$. Check graphically.

10. Show by the use of formula (7), Art. 124, that no tangent can be drawn to $xy = 8$ which has a positive slope.

11. Find the equations of all lines that are tangent to $x^2 + y^2 = 25$ and $x^2 + 4y^2 = 36$. Plot. *Ans.* $11y = \pm 4\sqrt{11}\,x \pm 15\sqrt{33}$.

12. Find the equation of a common tangent to $y^2 = 2px$ and $x^2 = 2py$. Check graphically.

13. Find the equation of the common tangent to $y^2 = 6x$ and $x^2 = 48y$. Check graphically.

14. Find the equations of tangents to $b^2x^2 - a^2y^2 = a^2b^2$ that intersect in the origin. *Ans.* The asymptotes, $bx - ay = 0,\ bx + ay = 0.$

15. For what value of m is $y = mx + 8$ tangent to $y^2 = 4x$? Plot.

16. A line is tangent to $x^2 + y^2 = 16$ and $y^2 = 6x$; find its equation. How many solutions? Plot.

17. Find the equations of lines of slope 2 which are tangent to

$$x^2 + y^2 - 4x + 6y + 5 = 0. \quad \text{Plot.}$$

18. Prove that $y - k = m(x - h) \pm r\sqrt{1 + m^2}$ is tangent to

$$(x - h)^2 + (y - k)^2 = r^2.$$

Suggestion. Move the origin to (h, k); use formula (1), Art. 124, and then translate the axes to the original position.

19. Find the equations of tangents to $x^2 + y^2 - 4x + 6y - 12 = 0$, with slope 2, by using the formula of Ex. 18.

20. Prove that $y - k = m(x - h) + \dfrac{p}{2m}$ is tangent to

$$(y - k)^2 = 2p(x - h).$$

21. Find the equation of a tangent to $y^2 - 2y - 4x = 0$ with a slope equal to 3.

22. Find the slope equation of a tangent to $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1.$

23. Find the equations of tangents to $4x^2 + 9y^2 + 8x - 36y + 4 = 0$ with slope equal to -3.

24. Find the equations of lines with slope equal to 2 which are tangent to $x^2 - y^2 = 1$.

25. Prove that a line with slope numerically less than $\dfrac{b}{a}$ cannot be tangent to $b^2x^2 - a^2y^2 = a^2b^2$.

26. Prove that any two tangents to $y^2 = 2px$ which are at right angles to each other intersect on the line $x = -\dfrac{p}{2}$, the directrix.

27. Show that any two tangents to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ which are perpendicular to each other intersect on the circle $x^2 + y^2 = a^2 + b^2$.

Suggestion. The equations of two tangents to the ellipse which are perpendicular to each other are

$$y = mx + \sqrt{m^2a^2 + b^2}, \tag{1}$$

and

$$y = \frac{x}{-m} + \sqrt{\frac{a^2}{m^2} + b^2}. \tag{2}$$

If these equations be regarded as simultaneous, the values of x and y that satisfy them are the coördinates of the intersection of the two tangents. If then m be eliminated between the two equations, an equation will be obtained which is satisfied by the coördinates of the intersection of *any* two perpendicular tangents.

In this case the elimination is easily made as follows: Eqs. (1) and (2) may be written

$$y - mx = \sqrt{m^2a^2 + b^2},$$

and

$$my - x = \sqrt{a^2 + m^2b^2}.$$

Then square and add.

28. Prove that the locus of foot of the perpendicular from the focus upon a tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is the circle $x^2 + y^2 = a^2$. Check graphically.

29. Show that any two tangents to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ which are perpendicular to each other intersect upon the circle $x^2 + y^2 = a^2 - b^2$, if $a > b$, but that there are no perpendicular tangents if $a < b$. What if $a = b$?

30. Prove that the locus of the foot of the perpendicular from the focus upon a tangent to $b^2x^2 - a^2y^2 = a^2b^2$ is the circle $x^2 + y^2 = a^2$. Check graphically.

31. Find the equation of the locus of the foot of the perpendicular from the center upon the tangent to $b^2x^2 + a^2y^2 = a^2b^2$.

Ans. $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

32. By transforming to polar coördinates reduce the equation of Ex. 31 to the form $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$.

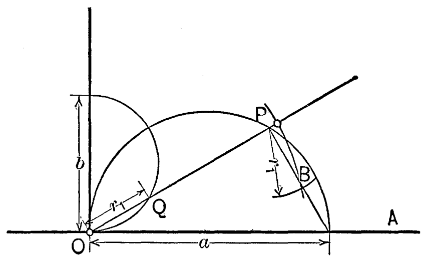


Fig. 121.

Construct the curve by use of the circles $r = a \cos\theta$, and $r = b \sin\theta$. (See Fig. 121.)

33. Find the equation of the locus of the foot of the perpendicular from the center upon a tangent to the equilateral hyperbola $x^2 - y^2 = a.^2$ *Ans.*

$(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

34. Show that the equation of the locus of Ex. 33 in polar coördinates is $r^2 = a^2 \cos 2\theta$. Plot the curve. This curve is called the **lemniscate**.

CHAPTER XI

SLOPES. TANGENTS AND NORMALS. DERIVATIVES

125. Introduction. In this chapter methods will be derived of finding the direction of a curve whose equation is known in rectangular coördinates at any point of the curve; of finding the equations of tangent and normal to the curve at any point; and some general methods established which will shorten the work of computing the slopes of curves. These methods will be shown in their application to some numerical cases.

126. Increments. In an equation connecting x and y, *e.g.*

$$4y = x^2 - 2x + 4, \tag{1}$$

if a value be assigned to x, y takes a value to correspond; and if x is given a different value, y will in general take a different value.

Thus, if $x = 0$, then $y = 1$; if $x = 1$, then $y = \frac{3}{4}$; if $x = -1$, then $y = \frac{7}{4}$; if $x = 2$, then $y = 1$.

Any change in x in general brings about a change in y.

These changes are most easily seen by referring to the curve which eq. (1) represents. As the point (x, y) traces the curve, both x and y change, and the amount that y changes depends upon the amount that x changes, and also upon the point of the curve from which the change is reckoned. Thus, if x increases by 1 from the value 1, y increases from $\frac{3}{4}$ to 1, or

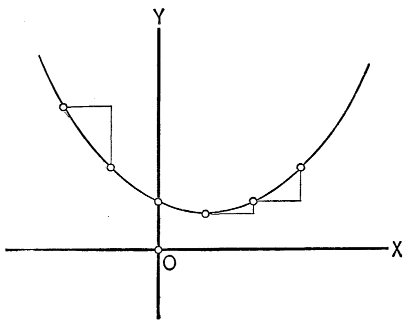


FIG. 122.

the increase in y is $\frac{1}{4}$; while if x increases by 1 from the value 2, y increases from 1 to $\frac{7}{4}$, or the increase in y is $\frac{3}{4}$. Again, if x increases from -3 to -2, y decreases from $\frac{19}{4}$ to $\frac{12}{4}$, or it may be said that the increase in y is $-\frac{7}{4}$.

Suppose now that some definite value of x is chosen, and a study made of the changes brought about in y by increasing x by small amounts from this definite value. Let the increase that is given to x be denoted by the symbol Δx, read "**delta** x," or "**increment** x"; and let the increase brought about in y by this change in x be denoted by Δy, read "**delta** y," or "**increment** y."

The following table shows values of x, y, Δx, Δy, and the ratio $\frac{\Delta y}{\Delta x}$, the value 2 being chosen for x from which to reckon the increments. The values of Δx are arbitrarily assumed.

$$4y = x^2 - 2x + 4.$$

x	y	Δx	Δy	$\frac{\Delta y}{\Delta x}$
2	1			
3	1.75	1	.75	.75
2.5	1.3125	.5	.3125	.625
2.1	1.0525	.1	.0525	.525
2.01	1.005025	.01	.005025	.5025
2.001	1.00050025	.001	.00050025	.50025
$2 + \Delta x$	$1 + \frac{\Delta x}{2} + \frac{\overline{\Delta x}^2}{4}$	Δx	$\frac{\Delta x}{2} + \frac{\overline{\Delta x}^2}{4}$	$.5 + \frac{\Delta x}{4}$

An examination of this table shows that as the increment in x is made smaller and smaller the corresponding increment in y becomes smaller and smaller, and approaches the limiting value zero when Δx approaches the limiting value zero. The ratio $\frac{\Delta y}{\Delta x}$, however, does not approach zero, but approaches the limiting value .5 when Δx approaches the limiting value 0.

127. Slope of the curve at any point. Look now at the geometric meaning of these facts. If P and P' denote the points $(2, 1)$ and $(2 + \Delta x, 1 + \Delta y)$ on the curve, then Δx and Δy have the values shown in the figure, and the ratio $\frac{\Delta y}{\Delta x}$ is the slope of the secant line through P and P'. As Δx approaches the limiting value zero, the point P' moves along the curve to the limiting position P, and the secant line through P and P' turns about P to the limiting position defined to be the tangent to the curve at P. Hence the slope of the tangent line at $(2, 1)$ is .5.

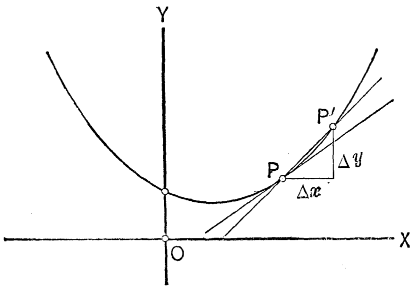


Fig. 123.

Definition. The slope of a tangent to a curve at any point is called the **slope of the curve** at that point.

The method here employed is a general one. By it one can compute the slope of the curve at any point. The table of values need not be computed, as in the preceding article, for this purpose.

E.g. to find the slope of the curve at the point where $x = 3$ one may proceed as follows:

Substitute $x = 3$ in the equation; then $y = \frac{7}{4}$.

Take a point on the curve near $P(3, \frac{7}{4})$. It may be represented by $P'(3 + \Delta x, \frac{7}{4} + \Delta y)$.

Since this point is on the curve, its coördinates must satisfy the equation of the curve.

$$\therefore\ 4\left(\tfrac{7}{4} + \Delta y\right) = (3 + \Delta x)^2 - 2(3 + \Delta x) + 4,$$

or $$\cancel{7} + 4\,\Delta y = \cancel{9} + 6\,\Delta x + \overline{\Delta x}^2 - \cancel{6} - 2\,\Delta x + \cancel{4},$$

or $$4\,\Delta y = 4\,\Delta x + \overline{\Delta x}^2.$$

$$\therefore\ \frac{\Delta y}{\Delta x} = 1 + \frac{\Delta x}{4}.$$

As Δx approaches the limiting value zero, *i.e.* as P' moves along the curve to coincide with P, the ratio $\dfrac{\Delta y}{\Delta x}$ approaches the limiting value 1, which is, therefore, the slope of the tangent line to the curve at the point $(3, \frac{7}{4})$.

EXERCISE XXXI

1. Compute the value of Δy when $\Delta x = .01$ for $x = .5$, 1, 10, respectively, in $y = x^3$.

2. Compute the slope of the curve $4y = x^2 - 2x + 4$ at the points where $x = -1$, $x = 0$, $x = 4$.

3. Find the slope of the curve $8y = x^3 + 1$ at the points where $x = 1$, 3, 0, -2.

4. Write the equation of the tangent line to the curve of eq. (1), Art. 126, at the point $(3, \frac{7}{4})$.

128. Equation of the tangent to a curve at any point. As a second example let it be required to find the slope of the tangent line to the curve

$$4x^2 + y^2 = 4 \qquad (1)$$

at any point (x_0, y_0) on the curve, and the equation of the tangent line at that point.

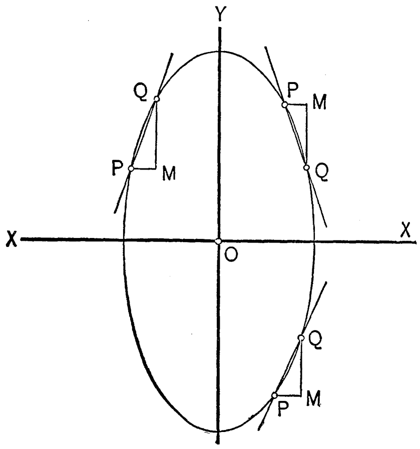


Fig. 124.

Let $P(x_0, y_0)$ be any point of the curve and Q $(x_0 + \Delta x, y_0 + \Delta y)$ a point of the curve near P. For convenience Δx is taken positive, and then Δy will be positive or negative according as the curve rises or falls toward the right from P. In the figure, for either position shown, $PM = \Delta x$, $MQ = \Delta y$.

Then $\frac{\Delta y}{\Delta x} =$ slope of the secant line PQ, and hence the limiting value of $\frac{\Delta y}{\Delta x}$, as Δx approaches the limiting value zero, is the slope of the tangent line to the curve at (x_0, y_0).

Since (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$ are points on the curve, the coördinates must satisfy eq. (1).

$$\therefore\ 4\,x_0^2 + y_0^2 = 4, \tag{2}$$

and

$$4\,(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 = 4. \tag{3}$$

Expanding eq. (3) and subtracting the corresponding members of eq. (2), there results

$$8\,x_0\,\Delta x + 4\,\overline{\Delta x}^2 + 2\,y_0 \Delta y + \overline{\Delta y}^2 = 0. \tag{4}$$

Every term of this equation contains either Δy or Δx as a factor. Take to the right member all the terms containing Δx and factor the two members of the equation. Then

$$\Delta y(2\,y_0 + \Delta y) = -\,\Delta x(8\,x_0 + 4\,\Delta x),$$

or

$$\frac{\Delta y}{\Delta x} = -\,\frac{8\,x_0 + 4\,\Delta x}{2\,y_0 + \Delta y}. \tag{5}$$

As Q moves along the curve to the limiting position P, both Δx and Δy approach the limiting value zero, and the right member of eq. (5) approaches the limiting value,

$$-\,\frac{4\,x_0}{y_0}.$$

Hence $-\,\frac{4\,x_0}{y_0}$ is the slope of the tangent line to the curve at (x_0, y_0).

Since the equation of a line of slope m through (x_0, y_0) is

$$y - y_0 = m(x - x_0),$$

therefore the equation of the tangent line to the curve at (x_0, y_0) is

$$y - y_0 = -\,\frac{4\,x_0}{y_0}(x - x_0). \tag{6}$$

This equation may be put into a simpler form as follows: Clear of fractions:

$$y_0y - y_0^2 = -4\,x_0x + 4\,x_0^2,$$

or

$$4\,x_0x + y_0y = 4\,x_0^2 + y_0^2.$$

The right member of this equation is, by eq. (2), equal to 4. Therefore

$$4\,x_0x + y_0y = 4 \tag{7}$$

is the equation of the tangent to

$$4\,x^2 + y^2 = 4,$$

at (x_0, y_0).

Since in eq. (7) (x_0, y_0) may be any point on the curve, the equation of the tangent line at any particular point may be written by substituting for x_0 and y_0 the coördinates of that point.

Thus the tangent at $(\frac{1}{2}, \sqrt{3})$, which is a point on the curve, is

$$2\,x + \sqrt{3}\,y = 4.$$

The student must not fail to recognize the fact that in eq. (7) x_0 and y_0 are the coördinates of a fixed point, the point of tangency, and that x and y are the variable coördinates of any point on the tangent line.

129. The normal. The normal to a curve at any point is the line perpendicular to the tangent at that point.

Since its slope is the negative reciprocal of the slope of the tangent, the equation of the normal to the curve of the preceding article at (x_0, y_0) is

$$y - y_0 = \frac{y_0}{4\,x_0}(x - x_0).$$

EXERCISE XXXII

Find the equations of tangents and normals to the following curves at the points assigned. Check graphically by plotting the curves and the lines whose equations are found.

1. $y^2 = 4\,x$ at $(1, -2)$.
2. $x^2 + y^2 = 25$ at $(-3, 4)$.

3. $y^2 = x^3$ at (x_0, y_0); at $(0, 0)$, $(1, 1)$, $(4, 8)$.

4. $y = mx + b$ at (x_0, y_0).

5. $y = x^2 + 4x - 5$ at the points where the curve crosses the x-axis.

6. $x^2 - y^2 = 16$ at $(5, 3)$.

7. $xy = 8$ at $(2, 4)$.

8. At what angles does the line $y = 3x + 2$ cut the parabola $y = x^2 + x - 6$? (By the angle between two curves is meant the angle between their tangents at the point of intersection.)

9. Find the angles at which $x^2 + y^2 = 25$ and $4x^2 = 9y$ intersect.

10. Find the point on the curve of example 5 where the slope is zero.

11. Find the point on the curve $y = -x^2 - 3x + 2$ where the slope is zero. Find also the point of the curve where the slope is 1. Where 2.

130. Tangent equations for reference. By the method used in Art. 128, the student can show that the following are the equations of the tangents to the given curves at the point (x_0, y_0).

Equation of Curve	Equation of Tangent
$x^2 + y^2 = r^2$.	$x_0x + y_0y = r^2$.
$y^2 = 2px$.	$y_0y = p(x + x_0)$.
$x^2 = 2py$.	$x_0x = p(y + y_0)$.
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.	$\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = 1$.
$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.	$\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = 1$.
$xy = c$.	$x_0y + y_0x = 2c$.

The student can more easily derive these equations after reading the remainder of this chapter.

DERIVATIVES. FORMULAS OF DIFFERENTIATION

131. Definitions and Notation. In the preceding articles the limiting value of $\dfrac{\Delta y}{\Delta x}$ as Δx approached the limiting value 0 at any point of the curve was found to represent the slope of the

tangent to the curve at that point. This limiting value of $\frac{\Delta y}{\Delta x}$ is of great importance in much advanced mathematical work, as well as for the study of curves. It is therefore worth while to assign a special name to this limiting value, and to develop short methods for computing it in given cases.

Definition. Given a function y, of a variable x, and a pair of corresponding values of x and y; if then an increment Δx be given to x, bringing about an increment Δy in y, the limiting value of $\frac{\Delta y}{\Delta x}$, as Δx approaches the limiting value zero, is called the **derivative of y with respect to x** for that value of x.

Notation. The symbol $\frac{dy}{dx}$ is used to denote the derivative of y with respect to x. The symbol $\left.\frac{dy}{dx}\right|_{x=x_0}$ means the value of that derivative for the value x_0 of x.

Thus in Art. 128, in the equation $4x^2 + y^2 = 4$,

$$\left.\frac{dy}{dx}\right|_{x=x_0} = -\frac{4x_0}{y_0},$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = -\frac{2}{\sqrt{3}}.$$

132. Geometric meaning of the derivative. A function y, of a variable x, may be represented graphically by a curve.

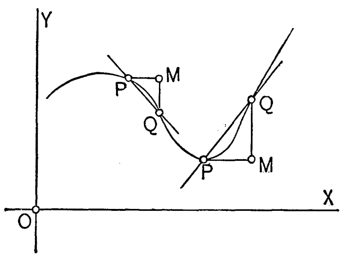


Fig. 125.

Let x_0 and y_0 be a pair of corresponding values of x and y. They are then the coördinates of some point on the curve. If an increment Δx be given to x, then y takes an increment Δy, as illustrated in the figure, where $PM = \Delta x$, $MQ = \Delta y$.

Then $\frac{\Delta y}{\Delta x}$ is the slope of the secant line through $P(x_0, y_0)$ and $Q(x_0 + \Delta x, y_0 + \Delta y)$.

Let Q move along the curve to the limiting position P; Δx and Δy both approach the limit 0, and the secant line approaches the limiting position of the tangent to the curve at P.

Hence the limiting value of $\frac{\Delta y}{\Delta x}$, as Δx approaches the limit 0, is the slope of the tangent to the curve at $P(x_0, y_0)$. Therefore,

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \textbf{the slope of the tangent to the curve at } (x_0, y_0).$$

The process of obtaining the derivative is called **differentiation.**

133. Continuity of functions. In the foregoing it was assumed that y is a single-valued, continuous function of x for all values of x under discussion. The meaning of this is explained in the following definition.

DEFINITION. A function y, of a variable x, is said to be a **single-valued** and **continuous** function for all values of x within an interval, if for each value of x in that interval there is a single, real, finite value of y, and if y changes gradually as x changes gradually, *i.e.* such that the change in y caused by a change in x, anywhere within the interval, can be made small at will by making the change in x small enough.

If y becomes infinite as x approaches a certain value as a limit, y is said to have an **infinite discontinuity** at that value of x.

If, as x passes through a certain value, y changes suddenly from one finite value to another, y is said to have a **finite discontinuity** at that value of x.

EXAMPLE 1. In $y = \frac{1}{x-2}$ as x approaches the limit 2 from

either side, y increases indefinitely in numerical value. Hence y has an infinite discontinuity at $x=2$.

EXAMPLE 2. In $y=\dfrac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1}$ if x is negative, but numerically very small, $2^{\frac{1}{x}}$ is very small and y is very near -1. Again, y may be written $y=\dfrac{1-2^{-\frac{1}{x}}}{1+2^{-\frac{1}{x}}}$, from which it is evident that y is very near 1 when x is positive and very small. Hence as x passes through 0 from negative to positive, y changes suddenly from -1 to $+1$. Therefore y has a finite discontinuity at $x=0$.

All, or nearly all, of the functions with which the student ordinarily deals are either continuous or have infinite discontinuities at definite points separated by finite intervals, and it will be assumed in what follows that the functions dealt with are finite and continuous for the values of the variable considered.

134. Formulas. In the following articles some general formulas of differentiation will be developed which will shorten the work of differentiation in certain cases.

135. Derivative of a constant. The derivative of a constant is zero:

$$\frac{dC}{dx}=0.$$

PROOF. Let C be any constant. Since C does not change as x changes by any amount Δx, the increment in C is zero; *i.e.* $\Delta C=0$.

$$\therefore \qquad \frac{\Delta C}{\Delta x}=0.$$

Therefore the limiting value of $\frac{\Delta C}{\Delta x}$ is zero,

or
$$\frac{dC}{dx} = 0.$$

This may also be seen geometrically by letting $y = C$. This is the equation of a straight line parallel to the x-axis. The value of $\frac{dy}{dx}$ at any point of this line is zero. (Art. 132.)

Hence $\frac{dy}{dx} = 0$, or since $y = C$, $\frac{dC}{dx} = 0$.

136. Derivative of a variable with respect to itself. The derivative of a variable with respect to itself is 1:

$$\boldsymbol{\frac{dx}{dx} = 1.}$$

Proof.
$$\frac{\Delta x}{\Delta x} = 1.$$

Therefore the limiting value of $\frac{\Delta x}{\Delta x}$ is 1.

$$\therefore \quad \frac{dx}{dx} = 1.$$

The student should illustrate this geometrically.

137. Derivative of a constant times a function. The derivative of a constant times a function is equal to the constant times the derivative of the function:

$$\boldsymbol{\frac{d\,(Cu)}{dx} = C\frac{du}{dx},}$$

where C is any constant and u is any function of x.

Proof. Let
$$y = Cu.$$

Let x take a particular value x_0. Then u and y take corresponding values u_0 and y_0, such that

$$y_0 = Cu_0.$$

Let x take an increment Δx; then u and y take increments Δu and Δy such that

$$y_0 + \Delta y = C(u_0 + \Delta u).$$

By subtraction, $\Delta y = C \cdot \Delta u.$

Divide by Δx; $\dfrac{\Delta y}{\Delta x} = C\dfrac{\Delta u}{\Delta x}.$

As Δx approaches the limit 0, $\dfrac{\Delta y}{\Delta x}$ and $\dfrac{\Delta u}{\Delta x}$ approach the limits $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ and $\left.\dfrac{du}{dx}\right|_{x=x_0}$.

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=x_0} = C \cdot \left.\frac{du}{dx}\right|_{x=x_0}.$$

Since x_0 is any value of x, then

$$\frac{dy}{dx} = C\frac{du}{dx},$$

or, since $y = Cu$,

$$\frac{d(Cu)}{dx} = C\frac{du}{dx}.$$

138. Derivative of a sum. The derivative of a sum of functions with respect to any variable is equal to the sum of the derivatives of the functions with respect to that variable:

$$\boldsymbol{\frac{d}{dx}(u + v + w \cdots) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \cdots}.$$

Proof. (For two functions.) Let u and v be two functions of x.

Let $y = u + v.$

Let $x = x_0$, then $y_0 = u_0 + v_0.$

Let $x = x_0 + \Delta x$, then

$$y_0 + \Delta y = u_0 + \Delta u + v_0 + \Delta v.$$

Subtracting, $\Delta y = \Delta u + \Delta v.$

Divide by Δx, $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Let Δx approach the limit 0;

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{du}{dx}\right|_{x=x_0} + \left.\frac{dv}{dx}\right|_{x=x_0},$$

or $$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

A similar proof holds for any number of functions.

139. Derivative of a product. The derivative of the product of two functions is equal to the sum of the products of each function times the derivative of the other:

$$\boldsymbol{\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}.$$

PROOF. Let u and v be any two functions of x.

Let $y = uv.$

Let $x = x_0$, then $y_0 = u_0 v_0.$

Let $x = x_0 + \Delta x$, then

$$y_0 + \Delta y = (u_0 + \Delta u)(v_0 + \Delta v).$$

Subtracting, $$\begin{aligned}\Delta y &= (u_0 + \Delta u)(v_0 + \Delta v) - u_0 v_0 \\ &= u_0 \Delta v + v_0 \Delta u + \Delta u \cdot \Delta v.\end{aligned}$$

Dividing by Δx, $$\frac{\Delta y}{\Delta x} = u_0 \frac{\Delta v}{\Delta x} + v_0 \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v.$$

Let Δx approach the limit 0; then Δu, Δv, and Δy each approaches the limit 0, and the limiting values $\frac{\Delta y}{\Delta x}$, $\frac{\Delta u}{\Delta x}$, and $\frac{\Delta v}{\Delta x}$ are, respectively, the derivatives of y, u, and v with respect to x for the value x_0.

$$\therefore \left.\frac{dy}{dx}\right|_{x=x_0} = u_0 \left.\frac{dv}{dx}\right|_{x=x_0} + v_0 \left.\frac{du}{dx}\right|_{x=x_0} + \left.\frac{du}{dx}\right|_{x=x_0} \cdot 0,$$

or, since x_0 is any value of x, and $y = uv$,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In a similar way a formula may be derived for the derivative of the product of three or more functions. However, one may make use of the formula just proved to obtain the derivative of the product of more than two functions. Thus,

$$\begin{aligned}\frac{d(uvw)}{dx} &= u\frac{d(vw)}{dx} + (vw)\frac{du}{dx} \\ &= u\left[v\frac{dw}{dx} + w\frac{dv}{dx}\right] + vw\frac{du}{dx} \\ &= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.\end{aligned}$$

140. Derivative of a quotient. The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerater, minus the numerator times the derivative of the denominator, divided by the square of the denominator:

$$\boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}.$$

Proof. Let $y = \dfrac{u}{v}$,

then, $vy = u$.

Differentiating, using the formula of the preceding article,

$$v\frac{dy}{dx} + y\frac{dv}{dx} = \frac{du}{dx}.$$

Solving for $\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{\frac{du}{dx} - y\frac{dv}{dx}}{v}.$

Replacing y by $\dfrac{u}{v}$,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

141. Derivative of the power of a function. The derivative of the nth power of a function is equal to n times the function to the power $n-1$, times the derivative of the function:

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}, \text{ where } n \text{ is constant.}$$

PROOF. (1) n a positive integer.

Let $$y = u^n.$$

Let $x = x_0$, then $$y_0 = u_0^n.$$

Let $x = x_0 + \Delta x$, then $$y_0 + \Delta y = (u_0 + \Delta u)^n.$$

Expanding $(u_0 + \Delta u)^n$ by the binomial theorem and subtracting,

$$\Delta y = nu_0^{n-1}\Delta u + \frac{n(n-1)}{1 \cdot 2}u_0^{n-2} \cdot \overline{\Delta u}^2 + \cdots + \overline{\Delta u}^n.$$

Every term on the right after the first contains Δu to a power higher than the first. Set out the factor Δu and divide both members by Δx:

$$\frac{\Delta y}{\Delta x} = \left[nu_0^{n-1} + \frac{n(n-1)}{2}u_0^{n-2}\Delta u + \cdots + \overline{\Delta u}^{n-1}\right]\frac{\Delta u}{\Delta x}.$$

Now as Δx approaches the limit 0, so do Δu and Δy. The limiting value of the quantity in the parenthesis is therefore nu_0^{n-1}.

$$\therefore \left.\frac{dy}{dx}\right|_{x=x_0} = nu_0^{n-1}\left.\frac{du}{dx}\right|_{x=x_0},$$

or

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

(2) n a negative integer. Let $n = -m$, where m is a positive integer.

Let $$y = u^n = u^{-m} = \frac{1}{u^m}.$$

Differentiate, using the formula for the derivative of a quotient,

$$\frac{dy}{dx} = \frac{u^m\dfrac{d1}{dx} - 1\dfrac{d(u^m)}{dx}}{u^{2m}}.$$

But $\frac{d1}{dx}=0$, by Art. 135, and since m is a positive integer,

$$\frac{d(u^m)}{dx}=mu^{m-1}\frac{du}{dx}, \text{ by part (1) of this article.}$$

$$\therefore \frac{dy}{dx}=\frac{-mu^{m-1}}{u^{2m}}\frac{du}{dx}$$

$$=-mu^{-m-1}\frac{du}{dx}$$

$$=nu^{n-1}\frac{du}{dx}, \text{ since } n=-m.$$

(3) n a rational fraction. Suppose $n=\frac{p}{q}$, where p and q are integers, either positive or negative.

Let $$y=u^n=u^{\frac{p}{q}}.$$

Raise both members of this equation to the qth power;

$$y^q=u^p.$$

Since both p and q are integers, the formula of this article may be applied.

$$\therefore qy^{q-1}\frac{dy}{dx}=pu^{p-1}\frac{du}{dx}.$$

$$\therefore \frac{dy}{dx}=\frac{pu^{p-1}}{qy^{q-1}}\frac{du}{dx}.$$

Now $$y^{q-1}=(u^{\frac{p}{q}})^{q-1}=u^{p-\frac{p}{q}}.$$

$$\therefore \frac{dy}{dx}=\frac{p}{q}\frac{u^{p-1}}{u^{p-\frac{p}{q}}}\frac{du}{dx}$$

$$=\frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}$$

$$=nu^{n-1}\frac{du}{dx}.$$

Hence $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$ if n is an integer or the ratio of two integers.

The proof can be extended to include irrational values of n, such as $\sqrt{2}$, π, etc., but it is not sufficiently elementary to be given here.

142. Summary. The above formulas are here collected and numbered for convenience of reference.

$$\text{I.} \qquad \frac{dC}{dx} = 0.$$

$$\text{II.} \qquad \frac{dx}{dx} = 1.$$

$$\text{III.} \qquad \frac{d(Cu)}{dx} = C\frac{du}{dx}.$$

$$\text{IV.} \qquad \frac{d}{dx}(u + v + w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}.$$

$$\text{V.} \qquad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\text{VI.} \qquad \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$\text{VII.} \qquad \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

143. Illustrations. Example 1. To find the derivative of $x^3 + 3x^2 + 5$ with respect to x.

$$\frac{d}{dx}(x^3 + 3x^2 + 5) = \frac{d(x^3)}{dx} + \frac{d(3x^2)}{dx} + \frac{d5}{dx}, \qquad \text{by IV,}$$

$$= 3x^2\frac{dx}{dx} + 3 \cdot 2x\frac{dx}{dx} + 0, \text{ by I, III, VII,}$$

$$= 3x^2 + 6x, \qquad \text{by II.}$$

Example 2. Given $z = 4t^3 + \sqrt{t^2+1}$; to find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{d(4t^3)}{dt} + \frac{d(t^2+1)^{\frac{1}{2}}}{dt}, \qquad \text{by IV,}$$

$$= 4 \cdot 3\,t^2 \cdot \frac{dt}{dt} + \frac{1}{2}(t^2 + t)^{-\frac{1}{2}}\frac{d(t^2+1)}{dt}, \quad \text{by III and VII,}$$

$$= 12\,t^2 + \frac{1}{2}(t^2+1)^{-\frac{1}{2}}\left(2\,t\frac{dt}{dt} + 0\right), \qquad \text{by II and IV,}$$

$$= 12\,t^2 + \frac{t}{\sqrt{t^2+1}}.$$

EXAMPLE 3. Given $pv = 4$; to find $\frac{dp}{dv}$.

Differentiate both members of the equation with respect to v;

$$\frac{d(pv)}{dv} = \frac{d(4)}{dv},$$

or

$$p\frac{dv}{dv} + v\frac{dp}{dv} = 0,$$

or

$$p + v\frac{dp}{dv} = 0.$$

$$\therefore \frac{dp}{dv} = -\frac{p}{v}.$$

EXAMPLE 4. Given the ellipse $4\,x^2 + y^2 = 16$; to find the slope of the tangent line at $(1, 2\sqrt{3})$.

The slope of the tangent line required is the value of $\frac{dy}{dx}$ at the point $(1, 2\sqrt{3})$. (Art. 132.)

Differentiating both members of the equation with respect to x,

$$\frac{d}{dx}(4\,x^2 + y^2) = \frac{d(16)}{dx}.$$

$$\therefore 8\,x\frac{dx}{dx} + 2\,y\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{4\,x}{y}.$$

Hence the slope of the tangent at $(1, 2\sqrt{3})$ is $-\frac{2}{\sqrt{3}}$.

The student should draw the ellipse and the line through $(1, 2\sqrt{3})$ with slope $-\frac{2}{\sqrt{3}}\cdot$

EXERCISE XXXIII

Find the derivative of each of the following functions with respect to its variable. The quantities a, b, c, m, n, are constants. All other letters represent variables.

1. $y = x^2 - \sqrt{x} - 3x + 5.$
2. $y = ax^2 + b + \frac{c}{x^2}\cdot$
3. $y = \sqrt{x^2 + a^2}.$
4. $s = \frac{t+a}{t-a}\cdot$
5. $y = x\sqrt{x^2+1}.$
6. $q = \sqrt{i^2 - a^2} + i.$
7. $z = t^2 - t^{-2}.$
8. $y = \sqrt{\frac{x^2-1}{x^2+1}}\cdot$
9. $y = (x+a)^n(x+b)^m.$
10. $z = \frac{a}{s^n}\cdot$
11. $y = (ax^2 + b)^n.$
12. $y = \left(\frac{x+a}{x-a}\right)^n\cdot$

Find the equations of the tangents to the following curves at the given points. Check by drawing the curves and the lines.

13. $y = mx + b$ at (x_0, y_0).
14. $y = 4x^2$ at $(1, 4)$.
15. $y^2 = 4x$ at $(1, 2)$.
16. $x^2 + y^2 = 25$ at $(-4, 3)$.
17. $x^2 - y^2 = 9$ at $(5, 4)$.
18. $xy = 8$ at $(2, 4)$.
19. $4x^2 + 16y^2 = 16$ at $(\sqrt{3}, \frac{1}{2})$.
20. $y = \frac{1}{(x-1)}$ at $(2, 1)$.
21. $y = ax^2 + bx + c$ at (x_0, y_0).
22. $x = ay^2 + by + c$ at (x_0, y_0).

23. Carefully construct the curve $pv = 4$, and by drawing tangents (approximately) at various points and measuring their slopes, verify the result found in example 3, Art. 143, viz. $\frac{dp}{dv} = -\frac{p}{v}\cdot$

24. Derive the equations of the tangents to the curves of Art. 130.

25. Show that the equation of the tangent to $ax^2+by^2+cx+dy+e=0$ at (x_0, y_0) is

$$ax_0x + by_0y + \frac{c}{2}(x + x_0) + \frac{d}{2}(y + y_0) + e = 0.$$

26. Show that the equation of the tangent to $y = x^3$ at (x_0, y_0) is

$$\frac{y + 2y_0}{3} = x_0^2x.$$

27. Show that the equation of the tangent to $y = ax^n$ at (x_0, y_0) is

$$\frac{y + (n-1)y_0}{n} = a{x_0}^{n-1}x.$$

28. Show that the equation of the tangent to

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \text{ at } (x_0, y_0) \text{ is}$$

$$ax_0x + \frac{b}{2}(x_0y + y_0x) + cy_0y + \frac{d}{2}(x + x_0) + \frac{e}{2}(y + y_0) + f = 0.$$

144. Limit of the ratio of a circular arc to its chord.

In Fig. 126 let BD and AD be tangents drawn at the ends of the circular arc AB. Then, since the arc of a circle is greater than its chord and less than any line which envelops it and has the same extremities,

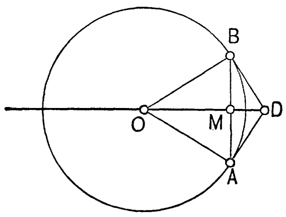


FIG. 126.

$$\text{chord } AB < \text{arc } AB < 2\,BD.$$

$$\therefore 1 < \frac{\text{arc } AB}{\text{chord } AB} < \frac{BD}{MB}.$$

Now let the point A move along the circle to the limiting position B. The line through A and B approaches the limiting position as tangent at B, and hence the angle MBD approaches the limit 0.

Hence $\dfrac{BD}{MB}$, which is equal to sec MBD, approaches the limit 1.

Therefore the ratio $\dfrac{\text{arc } AB}{\text{chord } AB}$ approaches the limit 1 as the arc approaches the limit 0; for it lies between 1 and a quantity whose limit is 1.

145. Circular or radian measure of an angle. The **radian** is defined to be the angle at the center of a circle whose arc is equal in length to the radius. Hence if the length of an arc of a circle be divided by the length of the radius, the quotient is the number of radians in the angle subtended at the center

by the given arc, or

$$\frac{\textbf{arc}}{\textbf{radius}} = \textbf{angle} \text{ (in radians).}$$

Hence in a circle of radius 1, "the arc equals the angle." That is, the number of linear units in the arc is equal to the number of radians in the subtended angle at the center.

146. Limit of $\frac{\theta}{\sin\theta}$. In Art. 144 if the circle has a radius equal to 1, and if the angle MOB is called θ, then chord $AB = 2\sin\theta$, and arc $AB = 2\theta$.

$$\therefore \frac{\text{arc } AB}{\text{chord } AB} = \frac{\theta}{\sin\theta}.$$

Therefore the ratio $\frac{\theta}{\sin\theta}$ approaches the limit 1 when θ approaches the limit 0.

147. Derivative of the sine.

Let $y = \sin u$, where u is a function of x.

In a circle of radius 1, let AOP be an angle at the center whose measure in radians is u. (Fig. 127.)

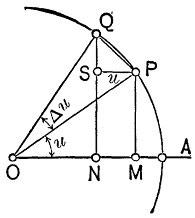


FIG. 127.

Then $MP = \sin u$. $\therefore MP = y$.

Let x take an increment Δx, bringing about an increment Δu in u, represented by the angle POQ.

Then arc $PQ = \Delta u$, and $SQ = \Delta y$.

In triangle PSQ,

$$\Delta y = \text{chord } PQ \cdot \sin SPQ.$$

$$\therefore \frac{\Delta y}{\Delta x} = \sin SPQ \cdot \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\Delta x},$$

or, since arc $PQ = \Delta u$,

$$\frac{\Delta y}{\Delta x} = \sin SPQ \cdot \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\Delta u}{\Delta x}.$$

Now as Δx approaches the limit 0 so also do Δu and Δy; the line through P and Q approaches the limiting position of the tangent at P, and hence the limiting value of SPQ is $\frac{\pi}{2} - u$.

Also the limiting value of $\dfrac{\text{chord } PQ}{\text{arc } PQ}$ is 1.

$$\therefore \frac{dy}{dx} = \sin\left(\frac{\pi}{2} - u\right)\frac{du}{dx},$$

or
$$\boldsymbol{\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}}.$$

148. Derivative of the cosine. In Fig. 127 let

$$z = \cos u;$$

then
$$OM = z,\ NM = -\Delta z.$$

$$\therefore\ -\Delta z = \cos SPQ \cdot \text{chord } PQ.$$

$$\therefore\ \frac{-\Delta z}{\Delta x} = \cos SPQ \cdot \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\Delta u}{\Delta x}.$$

Therefore, letting Δx approach the limit 0,

$$-\frac{dz}{dx} = \cos\left(\frac{\pi}{2} - u\right)\frac{du}{dx},$$

or
$$\boldsymbol{\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}}.$$

149. Derivatives of sine and cosine of an angle not in the first quadrant. The foregoing proofs have assumed the angle to be in the first quadrant. Proofs could as easily be given for the other quadrants, or they may be made to depend upon those above.

E.g. to find $\dfrac{d(\sin u)}{dx}$ for a value of u in the second quadrant.

Let $u = \frac{\pi}{2} + v$; then $\sin u = \cos v$, $\cos u = -\sin v$, and $\dfrac{du}{dx} = \dfrac{dv}{dx}$.

$$\therefore\ \frac{d(\sin u)}{dx} = \frac{d(\cos v)}{dx} = -\sin v \frac{dv}{dx}, \text{ by Art. 148,}$$

$$= \cos u \frac{du}{dx}, \text{ as before.}$$

Analytic proofs of the above formulas which are independent of the size of the angle are given in text-books on the Calculus.

150. Derivative of the tangent.

Let $$y = \tan u.$$

$$\therefore\ y = \frac{\sin u}{\cos u}.$$

Differentiating by the formula for a quotient,

$$\frac{dy}{dx} = \frac{\cos u \dfrac{d(\sin u)}{dx} - \sin u \dfrac{d(\cos u)}{dx}}{\cos^2 u}$$

$$= \frac{\cos u \cos u \dfrac{du}{dx} - \sin u\,(-\sin u)\dfrac{du}{dx}}{\cos^2 u}$$

$$= \frac{(\cos^2 u + \sin^2 u)\dfrac{du}{dx}}{\cos^2 u},$$

or $$\boldsymbol{\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}}.$$

151. Derivatives of cotangent, secant, cosecant. The student can show that the following formulas hold:

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}, \quad \frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx},$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}.$$

152. Summary. The formulas for the differentiation of the trigonometric functions are here collected and numbered consecutively with those of Art. 142.

$$\text{VIII.}\quad \boldsymbol{\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}}.$$

$$\text{IX.}\quad \boldsymbol{\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}}.$$

$$\text{X.}\quad \frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}.$$

$$\text{XI.}\quad \frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}.$$

$$\text{XII.}\quad \frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}.$$

$$\text{XIII.}\quad \frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}.$$

153. Illustrations. The foregoing formulas, together with those of Art. 142, enable one to find the derivative of any algebraic expression involving trigonometric functions. The following examples will help to make this clear.

EXAMPLE 1. Given $y = \sin^3 2x$; to find $\frac{dy}{dx}$.

By formula VII, $$\frac{dy}{dx} = 3 \sin^2 2x \frac{d(\sin 2x)}{dx}.$$

By VIII and II, $$\frac{d(\sin 2x)}{dx} = \cos 2x \frac{d(2x)}{dx}$$
$$= 2 \cos 2x.$$
$$\therefore \frac{dy}{dx} = 6 \sin^2 2x \cos 2x.$$

EXAMPLE 2. Given $z = \sqrt{1 + 2\tan^2 3s}$; to find $\frac{dz}{dt}$.

By VII, $$\frac{dz}{dt} = \frac{1}{2}(1 + 2\tan^2 3s)^{-\frac{1}{2}} \frac{d(1 + 2\tan^2 3s)}{dt}.$$

By IV, I, III, VII, and X,
$$\frac{d(1 + 2\tan^2 3s)}{dt} = 4 \tan 3s \cdot \sec^2 3s \cdot 3 \cdot \frac{ds}{dt}.$$
$$\therefore \frac{dz}{dt} = \frac{6 \tan 3s \sec^2 3s}{\sqrt{1 + 2\tan^2 3s}} \cdot \frac{ds}{dt}.$$

154. Other derivative formulas. Formulas for the derivatives of the inverse trigonometric functions, $\sin^{-1} u$, etc.; the loga-

rithmic functions, $\log_a u$; and the exponential functions, a^u, u^v, are derived in text-books on the Calculus.

The foregoing formulas will be sufficient for use in showing the application of derivatives to the study of curves, which is given in the next chapter.

EXERCISE XXXIV

Find the derivative of each of the following functions with respect to its variable:

1. $y = a\cos^2 x + b\sin^2 x.$
2. $y = 4\tan^3 2x.$
3. $z = \sin^{\frac{3}{2}} t.$
4. $s = \cos^2 x - \sin^2 x.$
5. $y = \cos^2(ax + b).$
6. $y = \dfrac{\sec^2 x}{\csc^2 x}.$
7. $y = \sec^4 3x.$
8. $s = \dfrac{\tan 2t}{1 + \sin t}.$
9. $y = \sin^2 x\sqrt{\sec x}.$
10. $y = \tan^n mx.$
11. $y = \sec^2 x - \csc^2 x.$
12. $y = \dfrac{1}{\sqrt{\sin x}}.$
13. $z = \frac{1}{3}\tan^3\theta - \tan\theta + \theta.$
14. $y = x\sin x.$
15. $y = x\tan x.$
16. $y = (\sin x)x - 1.$
17. $y = \cot 4x \csc 4x.$
18. $z = m\cot^n qx.$
19. $y = x(\sin x - \cos x).$
20. $q = a\sin^n bt.$

CHAPTER XII

MAXIMA AND MINIMA. DERIVATIVE CURVES

155. Maximum and minimum points of a curve. In the discussion that follows the curves are supposed to be such that the ordinate is a single-valued, continuous function of the abscissa. If the curve as a whole is not single valued, it can be divided into portions each of which is single valued.

For convenience, such a curve, or portion of a curve, may be thought of as generated by a point moving from left to right.

If, as the curve is so traced, the generating point rises to a certain position and then falls, that position is called a **maximum point** of the curve, and the ordinate at that point is called a **maximum ordinate.** If the generating point falls to a certain position and then rises, that position is called a **minimum point** of the curve, and the ordinate at that point a **minimum ordinate.**

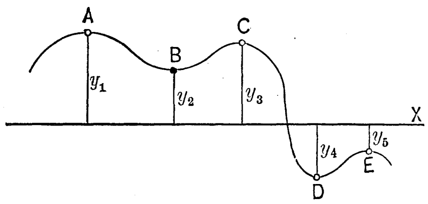


Fig. 128.

Thus A, C, and E are maximum points, and y_1, y_3, and y_5 are maximum ordinates, while B and D are minimum points, and y_2 and y_4 are minimum ordinates of the curve in Fig. 128.

According to the above definition a maximum ordinate is not necessarily the greatest ordinate of the curve. The definition requires only that a maximum ordinate shall be greater than the ordinates immediately to the right and left of it.

At a maximum point the curve is said to change from rising to falling, and at a minimum point to change from falling to rising.

156. Determination of the maximum and minimum points of a curve. The location of the maximum and minimum points of a curve whose equation in rectangular coördinates is known may be determined by use of the derivative. The method employed is a general one, but the solution of the equations is sometimes impossible. In the case of equations with numerical coefficients, however, an approximate solution can always be obtained.

It was shown in Art. 132 that $\frac{dy}{dx}$ for any point of the curve is equal to the slope of the tangent to the curve at that point. Then if $\frac{dy}{dx}$ is positive for a given point of the curve, the tangent line at that point makes with the x-axis an angle less than 90°, and hence the curve rises toward the right from that point. If $\frac{dy}{dx}$ is negative for a given point of the curve, the tangent at that point makes with the x-axis an angle between 90° and 180°, and hence the curve falls toward the right from that point.

Of course this rising or falling may continue for a very short distance only.

Figure 129 illustrates points of the curve for which $\frac{dy}{dx}$ is respectively positive, zero, and negative.

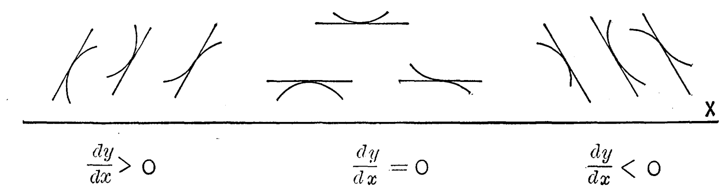
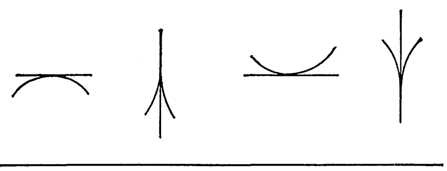


FIG. 129.

It follows from the above that if a point, in moving along the curve from left to right, passes through a position for which $\frac{dy}{dx}$ changes from positive to negative, the curve changes at that point from rising to falling, and hence that position is a maximum point of the curve; while if a point, in moving along the curve from left to right, passes through a position for which $\frac{dy}{dx}$ changes from negative to positive, such a position is a minimum point of the curve.

The derivative $\frac{dy}{dx}$ usually changes sign by passing through the value zero, so that the tangent at a maximum or minimum point of the curve is usually parallel to the x-axis. However, it may change sign by becoming infinite. In such a case the tangent is parallel to the y-axis at a maximum or minimum point. A point of this kind is called a **cusp-maximum**, or a **cusp-minimum**. (Fig. 130.)



Maximum Points. Minimum Points.

FIG. 130.

It does not follow, conversely, that if the tangent at a given point of the curve is parallel to one of the coördinate axes, the point is necessarily a maximum or minimum point. The curve may cross the tangent at that point. (See Fig. 129.)

The above discussion applies to only those parts of a curve for which neither coördinate becomes infinite. It frequently happens that as x passes through a certain value, $\frac{dy}{dx}$ changes sign, but neither a maximum nor minimum point of the curve corresponds to that value of x, because y there becomes infinite.

157. Illustration. To find the maximum and minimum

points of the curve

$$6y = 2x^3 - 3x^2 - 36x - 12. \tag{1}$$

Differentiating, $6\dfrac{dy}{dx} = 6x^2 - 6x - 36.$

$$\therefore \frac{dy}{dx} = x^2 - x - 6 \tag{2}$$

$$= (x+2)(x-3).$$

From this it is seen that if x has any value less than -2, both factors of $\dfrac{dy}{dx}$ are negative, and hence $\dfrac{dy}{dx}$ is positive. The curve therefore rises toward the right at all points for which $x < -2$.

If x is greater than -2 but less than 3, one factor of $\dfrac{dy}{dx}$ is positive and the other negative, and hence $\dfrac{dy}{dx}$ is negative. The curve therefore falls toward the right for all values of x between -2 and 3.

If $x > 3$, $\dfrac{dy}{dx}$ is positive, and hence the curve again rises toward the right for all values of $x > 3$.

As x passes through -2 from left to right, $\dfrac{dy}{dx}$ changes from positive to negative, and hence the point of the curve for which $x = -2$ is a maximum point; *i.e.* $(-2, 5\frac{1}{3})$ is a maximum point.

As x passes through 3 from left to right, $\dfrac{dy}{dx}$ changes from negative to positive, and hence $(3, -15\frac{1}{2})$ is a minimum point.

Figure 131 shows the curve plotted from these considerations and a few additional points through which it passes.

The meaning of the dotted curve is explained in the next article.

158. The first derivative curve. The facts of the preceding article are clearly brought out graphically by plotting the curve of eq. (2), using x as abscissa and $\frac{dy}{dx}$ as ordinate.

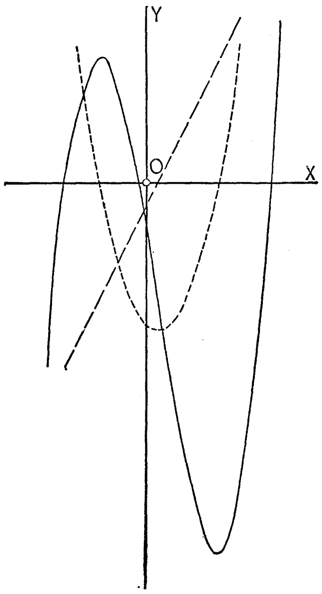


FIG. 131.

For convenience let $\frac{dy}{dx}$ be represented by z. Then eq. (2) becomes

$$z = x^2 - x - 6.$$

This curve, being a parabola, is easily plotted. It crosses the x-axis at -2 and 3, has its vertex at $(\frac{1}{2}, -\frac{25}{4})$, and its axis parallel to the y-axis (Art. 81). The locus is the dotted curve in Fig. 131.

The original curve will be referred to as the **primitive curve,** and the curve just described as the **first derivative curve.**

From the relations established in the preceding article, it follows that for those values of x for which the first derivative curve is above the x-axis, that is, z, or $\frac{dy}{dx}$, is positive, the primitive curve rises toward the right; for those values of x for which the derivative curve is below the x-axis, the primitive curve falls toward the right; for a value of x at which the first derivative curve crosses the x-axis from above, in going from left to right, the slope of the primitive curve changes from positive to negative, and hence the primitive curve has a maximum point; and for a value of x at which the first derivative curve crosses the x-axis from below in going from left to right, the primitive curve has a minimum point.

Moreover, the value of the ordinate of the first derivative curve gives one a good idea of the rapidity with which the primitive curve is rising or falling. Thus, if for a certain value of x, the ordinate of the first derivative curve is positive and numerically large, the primitive curve is rising rapidly toward the right for that value of x; while if the ordinate of the first derivative curve is negative and numerically small, for a certain value of x, the primitive curve is falling slowly toward the right for that value of x.

This is at once evident on remembering that the ordinate of the first derivative curve is equal to the slope of the primitive curve for the same value of x.

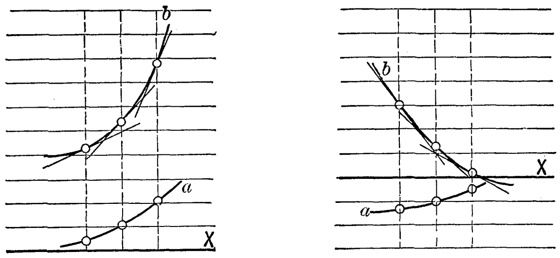
159. Concavity. Suppose that for $x = x_0$ the first derivative curve has a positive slope.

Let z_0, z_1, and z_2 be the ordinates of points on the derivative curve for the values x_0, $x_0 - \Delta x$, and $x_0 + \Delta x$ respectively, and let Δx be chosen small enough so that $z_1 < z_0 < z_2$.*

Then, since the values of z are equal to the slopes of the primitive curve for the same values of x, the tangent to the primitive curve must have turned counter-clockwise as x increased through x_0 from $x_0 - \Delta x$ to $x_0 + \Delta x$. This is true whether z_0 be positive, negative, or zero. (See Fig. 132.)

Exercise 1. In the curve $y = x^2 + 2x + 4$ draw the derivative curve, measure the ordinates at $x = 1\frac{1}{2}$, 2, $2\frac{1}{2}$, and draw the tangents to the primitive curve at points corresponding to the selected values of x. How would the tangent to the primitive curve turn as x increases through 2? Do the same for $x = -2\frac{1}{2}$, -2, $-1\frac{1}{2}$.

* This is possible since the slope of a curve at any point is the limiting value of the slope of a secant line through that point and a neighboring point of the curve. The secant line, cutting either to the right or left of the given point, can then be brought near enough to the tangent to have a positive slope, since the slope of the tangent is positive. The ordinates of the curve are therefore greater just to the right and less just to the left than the ordinate at the point of tangency.



a Derivative Curve. *b* Primitive Curve.

Fig. 132.

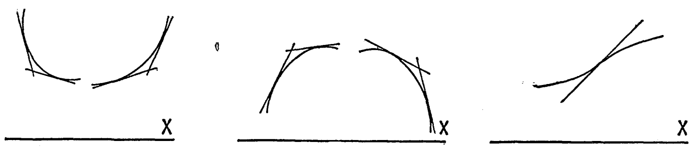
Exercise 2. Prove that if the derivative curve has a negative slope for $x = x_0$, the tangent to the primitive curve turns clockwise as x increases through x_0.

Exercise 3. Illustrate the law stated in exercise 2 by using the curve $y = -x^2 + 2x - 3$.

Definitions. If the tangent to a curve turns counterclockwise as the point of tangency moves to the right through a given point, the curve is said to be **concave up** at that point; while if the tangent turns clockwise as the point of tangency moves to the right through a given point, the curve is said to be **concave down** at that point.

A point on the curve where the curve changes from concave up to concave down, or *vice versa,* is called a **point of inflexion.**

As the point of tangency passes through a point of inflexion, the tangent line changes the direction of rotation. The curve crosses the tangent at a point of inflexion.



Slope increasing toward the right. Curve concave up.

Slope decreasing toward the right. Curve concave down.

Point of inflexion.

Fig. 133.

The results of this article may be stated as follows: For all values of x for which the first derivative curve is rising toward the right, the primitive curve is concave upward; for all values of x for which the first derivative curve is falling toward the right, the primitive curve is concave downward; for a value of x for which the first derivative curve has a maximum or minimum point, the primitive curve has a point of inflexion.

160. The second derivative. The derivative of a function of a variable is itself a function of that variable. This derivative may then also be differentiated.

Thus, if
$$y = 2x^3 + \sin 2x,$$
$$\frac{dy}{dx} = 6x^2 + 2\cos 2x,$$
and
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 12x - 4\sin 2x.$$

The derivative, $\frac{dy}{dx}$, is called the **first derivative** of y with respect to x, and $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the **second derivative** of y with respect to x.

The symbol $\boldsymbol{\frac{d^2y}{dx^2}}$ is used to denote the second derivative of y with respect to x, thus $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

Similarly, $\frac{d^3y}{dx^3}$ means $\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$, etc.

161. The second derivative curve. The second derivative is related to the first derivative in precisely the same way as the first derivative is related to the primitive function. But it also has an interesting and important relation to the primitive function, now to be explained.

Suppose the second derivative to be represented by a curve, using x for abscissa and $\frac{d^2y}{dx^2}$ as ordinate. This curve is called the **second derivative curve.**

Then, for all values of x for which the second derivative curve is above the x-axis, the primitive curve is concave up; for the ordinate of the second derivative curve is equal to the slope of the first derivative curve for the same value of x, and where the slope of the first derivative curve is positive, the primitive curve is concave up. (Art. 159.)

In like manner it is proved that for those values of x for which the second derivative curve is below the x-axis, the primitive curve is concave down.

For a value of x at which the second derivative curve crosses the x-axis, the first derivative curve has either a maximum or minimum point, and hence the primitive curve has a point of inflexion. (Art. 159.)

162. Summary. The results of the foregoing discussion of this chapter may be summarized as follows:

For all values of x for which the first derivative curve is above the x-axis, the primitive curve rises toward the right; for all values of x for which the first derivative curve is below the x-axis, the primitive curve falls toward the right; for a value of x at which the first derivative curve crosses the x-axis from above in going from left to right, the primitive curve has a maximum point; for a value of x at which the first derivative curve crosses the x-axis from below in going from left to right, the primitive curve has a minimum point.

For all values of x for which the second derivative curve is above the x-axis, the primitive curve is concave up; for all values of x for which the second derivative curve is below the x-axis, the primitive curve is concave down; for a value of x at which the second derivative curve crosses the x-axis, the primitive curve has a point of inflexion.

163. Illustrations. Example 1. Given

$$y = \sin x + 3,$$

then $$\frac{dy}{dx} = \cos x,$$

and $$\frac{d^2y}{dx^2} = -\sin x.$$

The curves are shown in Fig. 134, and the relations established above are seen to hold.

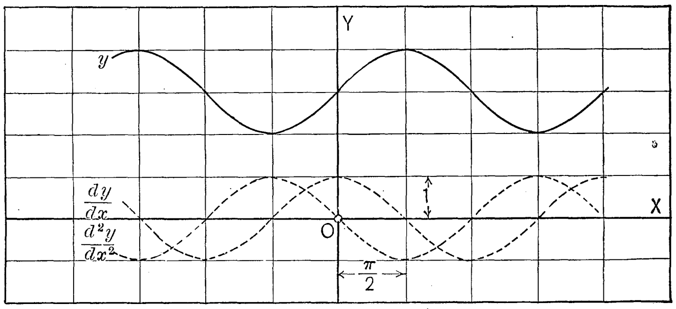


Fig. 134.

The student should make a careful study of the figure.

Example 2. As another illustration, study the curves of Fig. 131. The straight line in the figure represents the equation

$$\frac{d^2y}{dx^2} = 2x - 1.$$

Example 3. A circular cistern is to be built to have a given capacity; to find its dimensions in order that the amount of lining required will be a minimum.

Let H = depth, D = diameter, and S = area of inner surface.

Then $$S = \frac{\pi D^2}{4} + \pi DH.$$

But $$\text{vol.} = \frac{\pi D^2 H}{4} = C, \text{ where } C \text{ is constant.}$$

$$\therefore \quad S = \frac{\pi}{4} D^2 + \frac{4C}{D}.$$

Here S is expressed as a function of the variable D. From the equation it is at once evident that if D is very small the surface is very large, and is again very large when D is very large; while for intermediate values of D the surface has smaller values. The curve which represents the equation between S and D therefore falls and then rises as D increases from 0, as in Fig. 135. There will therefore be a minimum point, which may be found by equating to 0 the value of $\frac{dS}{dD}$.

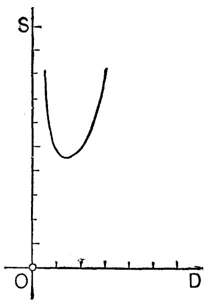


FIG. 135.

$$\frac{dS}{dD} = \frac{\pi}{2} D - \frac{4C}{D^2}.$$

Equating this expression to 0, and solving for D,

$$D = \sqrt[3]{\frac{8C}{\pi}}.$$

The relation between D and H is most easily obtained by replacing C by $\frac{\pi D^2 H}{4}$ in the expression for $\frac{dS}{dD}$, and equating the result to 0. Then

$$\frac{\pi}{2} D - \frac{4 \pi D^2 H}{4 D^2} = 0,$$

or

$$D = 2H.$$

EXERCISE XXXV

Sketch the following curves, first sketching the first and second derivative curves. Locate maximum and minimum points and points of inflexion.

1. $y = x^2 - 4x + 5$.
2. $y = -x^2 - x + 3$.
3. $3y = x^3 - 12x + 6$.
4. $y = \sin 2x$.
5. $y = \cos\left(x - \frac{\pi}{6}\right)$.
6. $6y = 2x^3 - 3x^2 - 12x - 6$.
7. $10y = 2x^3 + 9x^2 - 24x + 20$.

8. $20\,y = x^3 - 9\,x^2 + 15\,x - 20.$

9. $y = x^3 - 3\,a^2x + b^3.$

10. $y = x^4.$

11. $y = x(a - x).$

12. $y = ax^2 + bx + c.$

13. $y = \sin x + a.$

14. $y - 1 = (x - 2)^3.$

15. $y = \dfrac{x^3}{x - 4}.$

16. $3\,y = x^3 - 3\,x^2 + 6\,x - 1.$

17. $12\,y = 3\,x^4 - 8\,x^3 - 30\,x^2 + 72\,x + 24.$

18. $8\,y = x^4 - 6\,x^2 + 8\,x + 16.$

19. In $y = ax^3 + bx + c$, where $a \neq 0$, show that there is a maximum and a minimum point if b and a are opposite in sign, but that there is neither maximum nor minimum if a and b are of the same sign, or if $b = 0$.

Compare the curves obtained by using the following values of a, b, and c. (1) $a = 1$, $b = -3$, $c = 2$; (2) $a = 1$, $b = -.03$, $c = 2$; (3) $a = 1$, $b = -.0003$, $c = 2$. If $a > 0$, and a and c are held fast while b is made to approach the limit 0 from the negative side, what becomes of the maximum and minimum points? If b then becomes positive, how is the tangent at the point of inflexion affected?

20. In $y = ax^3 + bx^2 + cx + d$ show that there is a maximum and a minimum point if $b^2 - 3\,ac > 0$, but not otherwise. How does the case where $b^2 - 3\,ac = 0$ differ from that where $b^2 - 3\,ac < 0$?

21. The equation of the path of a projectile, fired at an angle α to the horizontal with an initial velocity V, is

$$y = x \tan \alpha - \frac{gx^2}{2\,V^2 \cos^2 \alpha}.$$

Find the maximum height to which the projectile rises. *Ans.* $\dfrac{V^2 \sin^2 \alpha}{2\,g}$.

22. Letting $R =$ the range on the horizontal of the projectile described in ex. 21, show that $R = \dfrac{V^2 \sin 2\,\alpha}{g}$.

Letting α vary, plot the curve which represents R as a function of α. For what value of α is R a maximum? *Ans.* $\dfrac{\pi}{4}$.

23. Prove that the greatest rectangle of a given perimeter is a square.

24. A cylindrical tin can, closed at both ends, is to be made to have a certain capacity. Show that the amount of tin used will be a minimum when the height equals the diameter.

25. Show that the rectangle of greatest area that can be inscribed in a circle is a square.

26. Given that the strength of a rectangular beam of given length varies as the product of the breadth and the square of the depth, find the ratio of depth to breadth of the strongest beam that can be cut from a cylindrical log. *Ans.* $h = \sqrt{2} \cdot b$.

27. Given that the deflection, under a given load, of a rectangular beam of given length, varies inversely as the product of the breadth and the cube of the depth, find the ratio of depth to breadth of the beam of least deflection that can be cut from a cylindrical log. *Ans.* $h = \sqrt{3} \cdot b$.

SUGGESTION. Make the reciprocal of the deflection a maximum.

28. A rectangular piece of tin of width b is to be bent up at the sides to form an open trough of rectangular cross section. Find the width of the strip bent up at each side when the carrying capacity is a maximum.

Ans. $\frac{b}{4}$.

29. Find the dimensions of the greatest right circular cylinder, the sum of the length and girth of which is 6 ft.

Ans. $H = 2$ ft., Diam. $= \frac{4}{\pi}$ ft.

30. Find the dimensions of the greatest rectangular box of square base, the sum of the length and girth of which is 6 ft. *Ans.* Length $= 2$ ft.

31. Find the ratio of altitude to radius of base of the conical vessel, of open base, which requires the least amount of material for a given capacity.

Ans. Alt. $= \sqrt{2}$ rad.

32. A point moves along a straight line. At the time t its distance from a fixed point of the line is s: at the time $t + \Delta t$, its distance is $s + \Delta s$. Then $\frac{\Delta s}{\Delta t}$ is the average velocity of the point for the time Δt. The limiting value of $\frac{\Delta s}{\Delta t}$, as Δt approaches 0 as a limit, is defined to be the velocity, v, at the time t. Hence $v = \frac{ds}{dt}$.

Given $s = 16\,t^2$, find the velocity at any time t.

33. The average acceleration, during an interval of time, of a point moving in a straight line, is the increase in velocity during that time, divided by the length of the interval of time.

Make a definition for the acceleration *at* any instant, and show that the acceleration is

$$\frac{dv}{dt}, \text{ or } \frac{d^2s}{dt^2}.$$

Find the acceleration if $s = 16\,t^2$.

34. Plot the curves representing the space, velocity, and acceleration, in terms of the time, if $s = 16\,t^2$.

35. Given $s = at^2 + bt + c$, where a, b, and c are constant, show that the velocity in terms of the time is represented by a straight line, and that the acceleration is constant.

36. The formula for the space traversed by a body projected vertically upward, with velocity v_0, is

$$s = v_0 t - 16\,t^2 \qquad (s \text{ in ft., } t \text{ in secs.})$$

Find, by differentiation, the velocity and acceleration of a bullet fired upward with initial velocity of $1000\,f/s$.

Plot the curves representing space, velocity, and acceleration in terms of the time. How high does the bullet go ?

37. A point moves back and forth along a diameter of a circle of radius a, with simple harmonic motion (Art. 116), making n complete oscillations per unit of time. If s is the abscissa of the point referred to the center, and the point is at the end of the diameter when $t = 0$, show that

$$s = a \cos(2\,\pi n t).$$

Find also the velocity and acceleration at any time, and plot the curves for space, velocity, and acceleration.

38. Since $\frac{d}{dx}(x^2 + C)$ is the same as $\frac{d}{dx}(x^2)$, how many primitive curves are there whose first derivative curve is

$$\frac{dy}{dx} = 2\,x\,?$$

Sketch some of the derivative curves. How are they situated with reference to each other ? What is the equation of the primitive which passes through (2, 5) ?

39. Find the primitives of which $\frac{dy}{dx} = \cos x$ is the first derivative curve.

40. Find the primitive of which $\frac{d^2y}{dx^2} = 2$ is the second derivative curve, and which passes through (4, 1) with a slope equal to 3.

41. Show that for the second derivative curve $\frac{d^2y}{dx^2} = a$, a primitive may be obtained which passes through any given point in any given direction.

CHAPTER XIII

THE CONIC SECTIONS

164. Definition of the conic. A **conic section**, or simply **conic**, is the curve of intersection of the surface of a right circular cone and a plane. It can be shown, however, that the following definition is equivalent to the one just given.

DEFINITION. A **conic** is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point in the plane to its distance from a fixed straight line in the plane is constant.

This definition will be adopted here.

The fixed point is called the **focus**, the fixed straight line the **directrix**, and the constant ratio the **eccentricity**, of the conic.

165. Construction of conics. Let F be the focus, DD' the directrix, and e the eccentricity. Let P be any point on the conic, and M the foot of the perpendicular drawn from P to the directrix. Then, by definition of the conic,

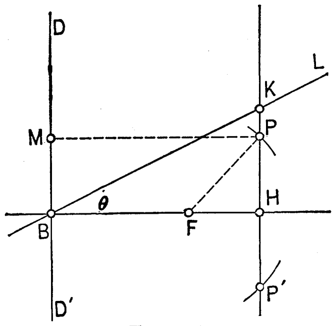


FIG. 136.

$$\frac{FP}{MP} = e.$$

(The lines FP and MP are to be counted as positive, whatever their direction.)

This suggests the following method of locating points of the conic: Through F draw a line FB perpendicular to DD', intersecting DD' in B. Through B draw a line BL, making an angle θ with BF such that

$\tan\theta = e$. Take any point H on BF, and let the perpendicular to BF through H meet BL in K. Then, $\frac{HK}{BH} = \tan\theta = e$. With F as a center and a radius equal to HK, describe an arc of a circle cutting HK in P and P'. The points P and P' so obtained are points on the conic.

In this manner, as many points as desired may be obtained, and the conic sketched by drawing a smooth curve through them.

Evidently they lie in pairs which are symmetrical with FB as an axis of symmetry. This line FB is called the **axis** of the conic.

166. Vertices of a conic. The points of the conic which lie on the line through the focus perpendicular to the directrix are called the **vertices** of the conic.

To obtain these points, draw lines through F inclined 45° and 135° to the line BF. From the points of intersection of these lines with BL drop perpendiculars to BF. The feet of these perpendiculars are the vertices, as the student can easily show.

If $e = 1$, there is only one vertex, but if $e \neq 1$, there are two vertices.

The figures on the following pages show conics constructed for $e = \frac{4}{5}$, $e = 1$, and $e = \frac{3}{2}$.

EXERCISE XXXVI

1. Plot in different figures the conics for $e = \frac{1}{2}$, $e = 1$, $e = \frac{4}{3}$.

2. Plot in the same figure, using the same directrix and focus for all the curves, the conics for $e = .9$, $e = 1$, $e = 1.1$.

3. Assume a unit of distance, and taking the distance from focus to directrix to be 1, 2, 4, 20, respectively, construct the conics for $e = 1$.

4. Same as example 3 for $e = \frac{2}{3}$.

5. Same as example 3 for $e = \frac{3}{2}$.

6. Prove that the conic is tangent to the line BL at the intersection of BL and a line through F parallel to the directrix.

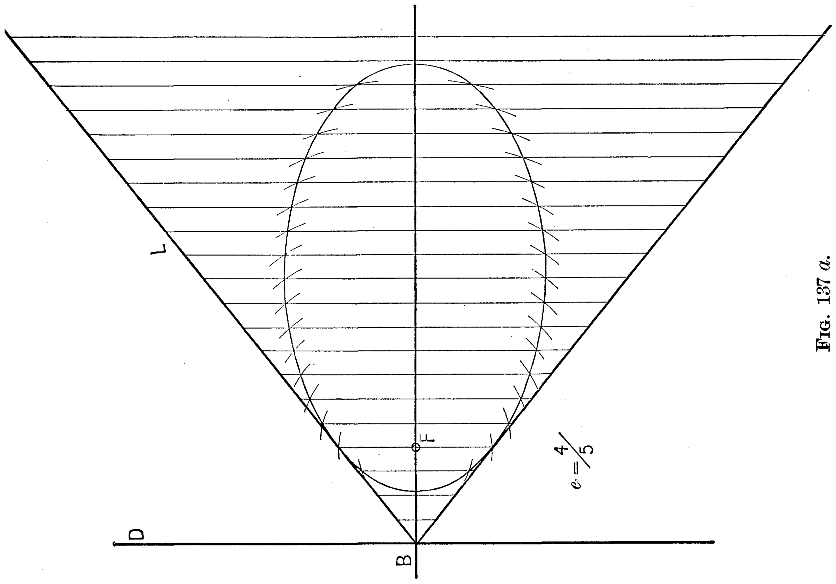


FIG. 137 *a.*

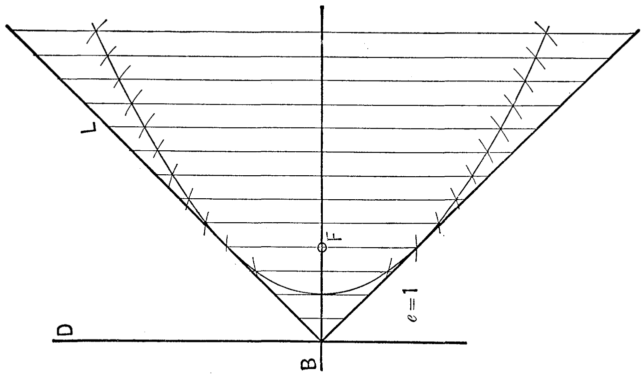


FIG. 137 *b.*

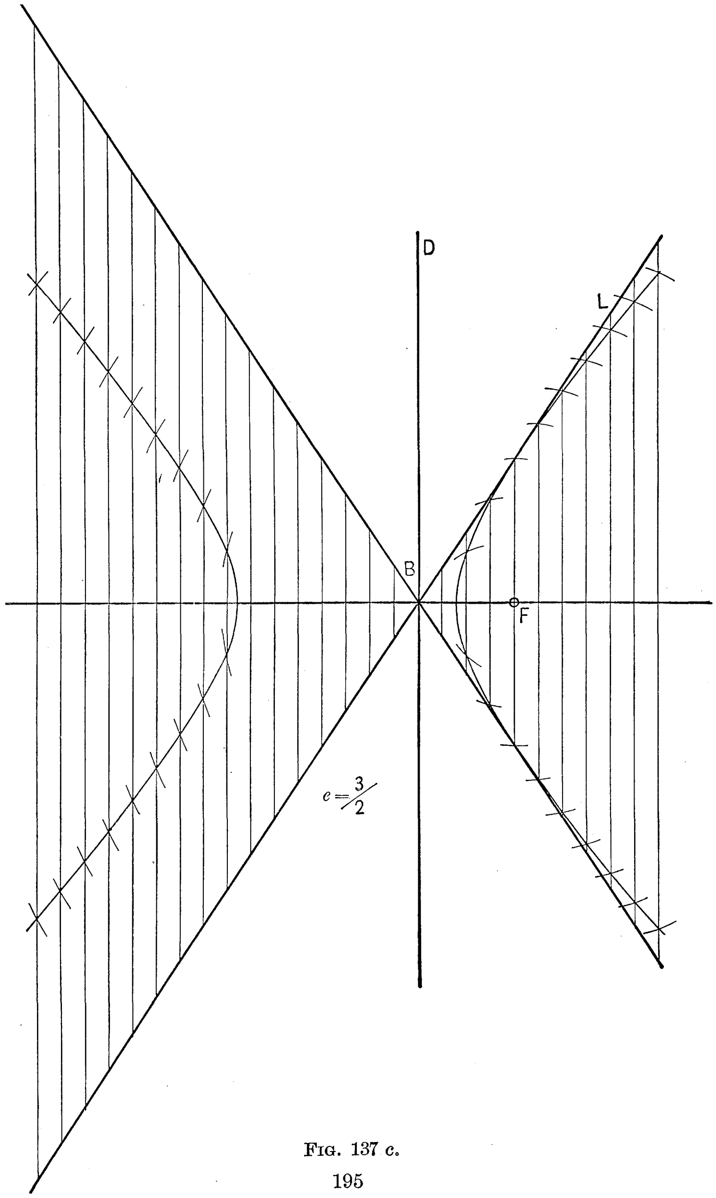


Fig. 137 c.

167. Classification of conics. From the constructions already made, it is evident that the general shape of the conic depends upon the value of e, and that the conics may be divided into three classes, according as $e < 1$, $e = 1$, or $e > 1$.

A conic whose eccentricity is less than 1 is an **ellipse**; one of eccentricity equal to 1, a **parabola**; and one of eccentricity greater than 1, an **hyperbola**. (See footnote, Art. 171.)

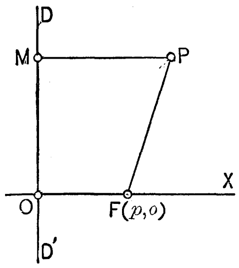


FIG. 138.

168. The equation of the conic in rectangular coördinates. Let the directrix be taken as y-axis and the line through the focus perpendicular to the directrix as the x-axis. Let the distance from the directrix to focus be p. Then the coördinates of F are $(p, 0)$. Let $P(x, y)$ be any point on the conic, and MP the distance from P to the directrix. Then, from the definition of the conic,

$$\frac{FP}{MP} = e, \text{ or } FP = eMP.$$

But $$FP = \sqrt{(x-p)^2 + y^2}, \text{ and } MP = x.$$

$$\therefore \quad (x-p)^2 + y^2 = e^2x^2,$$

or $$\mathbf{(1 - e^2)x^2 - 2\,px + y^2 + p^2 = 0.}$$

This is, therefore, the equation of any conic when the y-axis is the directrix and the x-axis is the line through the focus perpendicular to the directrix.

169. The parabola. $e = 1$. In the equation just found let $e = 1$. The conic is then a parabola. The equation reduces to

$$y^2 = 2\,px - p^2.$$

This equation of the parabola was obtained in Art. 75, and from the same definition as here used. The equation was discussed in that place. The student should review Arts. 75–78 at this time.

170. The centric conics. $e \neq 1$. In the equation of Art. 168,

$$(1-e^2)x^2 - 2px + y^2 + p^2 = 0,$$

divide by the coefficient of x^2 and then complete the square in the terms containing x,

$$x^2 - \frac{2p}{1-e^2}x + \frac{p^2}{(1-e^2)^2} + \frac{y^2}{1-e^2} = \frac{p^2}{(1-e^2)^2} - \frac{p^2}{1-e^2} = \frac{p^2e^2}{(1-e^2)^2},$$

or
$$\left(x - \frac{p}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{p^2e^2}{(1-e^2)^2}.$$

Substitute
$$x' = x - \frac{p}{1-e^2}, \quad y' = y,$$

which transforms to parallel axes through $\left(\frac{p}{1-e^2}, 0\right)$. (Art. 52.) The equation then becomes

$$x'^2 + \frac{y'^2}{1-e^2} = \frac{p^2e^2}{(1-e^2)^2}.$$

Dividing by the right-hand member brings the equation into the form

$$\frac{x'^2}{\frac{p^2e^2}{(1-e^2)^2}} + \frac{y'^2}{\frac{p^2e^2}{1-e^2}} = 1. \tag{A}$$

Since this equation contains only even powers of x and y, the curve is symmetric with respect to both coördinate axes, and hence with respect to the origin. The origin may therefore be called the **center** of the conic, and the conic called a **centric conic**.

Also, since the conic is symmetric with respect to the center, rotation of the conic in its own plane through 180° about its center will bring the conic back into its original position, having merely interchanged the points. Let the conic, together with its focus and directrix, be thus rotated. The focus and directrix are brought into new positions which are symmetric with respect to the center. They have remained focus and directrix of the conic, however, and since the new position is

the same as the old position they must be focus and directrix of the conic in its original position.

Therefore every centric conic has two foci and two directrices.

They are respectively symmetric with respect to the center.

171. The ellipse. $e<1$. In eq. (A) of the preceding article, the divisors of x'^2 and y'^2 are both positive if $e<1$. For convenience let

$$a^2=\frac{p^2e^2}{(1-e^2)^2},\quad b^2=\frac{p^2e^2}{1-e^2}. \tag{1}$$

Substituting these values in eq. (A) and dropping primes, it becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This is known as the standard form of the equation of the ellipse.*

172. Axes of the ellipse. Letting $y=0$, the intercepts of the ellipse on the x-axis are found to be a and $-a$. The intercepts on the y-axis are b and $-b$.

The length $2a$ is called the **major axis**, and $2b$ **the minor axis.**

The relation connecting a, b, and e is found from eq. (1) of the preceding article to be

$$a^2(1-e^2)=b^2.$$

This equation shows that $a>b$.

* In Art. 83 the ellipse was defined in an altogether different way. The equation of the ellipse derived from that definition and that just derived are, however, the same, which proves that the two definitions are equivalent. The property of the ellipse used in Art. 83 as a definition will be shown in a succeeding article to follow from the definition used in this chapter.

A like remark applies to the hyperbola.

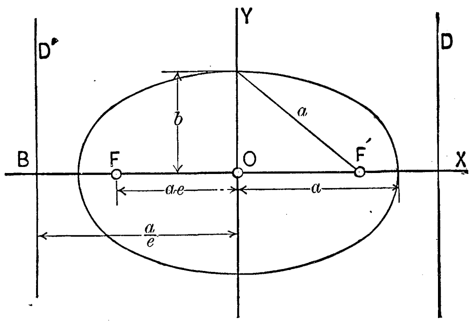


FIG. 139.

The abscissa of the new origin referred to the old in the transformations of Art. 170 is $\frac{p}{1-e^2}$; *i.e.*

$$BO = \frac{p}{1-e^2}.$$

Now $$a = \frac{pe}{1-e^2}.$$

$$\therefore\ BO = \frac{a}{e}.$$

Also $$FO = BO - BF = \frac{p}{1-e^2} - p = \frac{pe^2}{1-e^2},$$

or $$FO = ae.$$

The relation $a^2(1-e^2) = b^2$ may be written $a^2e^2 = a^2 - b^2$, from which

$$\boldsymbol{ae = \sqrt{a^2 - b^2}}.$$

Therefore if the end of the minor axis be taken as a center and an arc described with the semi-major axis as a radius, this arc will cut the major axis in the focus.

173. Summary. In an ellipse whose major axis is $2\,a$, minor axis $2\,b$, and eccentricity e, the following relations hold:

$\boldsymbol{a^2e^2 = a^2 - b^2}$.

$\boldsymbol{ae}$ = **distance from center to focus,**

$\boldsymbol{\frac{a}{e}}$ = **distance from center to directrix.**

174. The hyperbola. $e > 1$. In eq. (A), Art. 170, the divisor of y'^2 is negative if $e > 1$. Let then

$$a^2 = \frac{p^2 e^2}{(1-e^2)^2}, \quad b^2 = \frac{p^2 e^2}{e^2 - 1}.$$

Then both a and b are real.

Substituting these values in eq. (A) and dropping primes, it becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is known as the standard form of the equation of the hyperbola.

(See also Art. 87, and the footnote to Art. 171.)

175. Axes of the hyperbola. Letting $y = 0$, the intercepts on the x-axis are seen to be a and $-a$. If $x = 0$, y is imaginary.

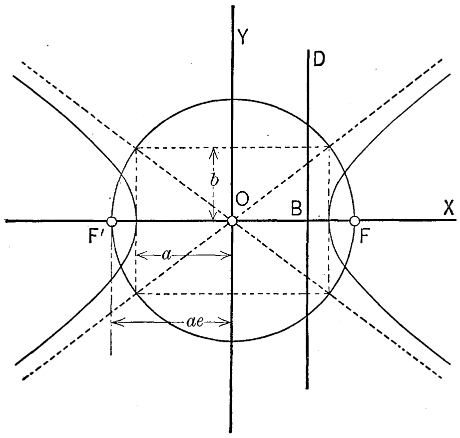


FIG. 140.

Hence the curve does not cross the y-axis.

The length $2a$ is called the **transverse axis**, and $2b$ the **conjugate axis**.

The relation connecting a, b and e is $b^2 = a^2(e^2 - 1)$, or

$$a^2 e^2 = a^2 + b^2.$$

This shows that $b \gtreqless a$ according as $e \gtreqless \sqrt{2}$.

As in the ellipse, the abscissa of the center referred to the old origin, on the directrix, is $\frac{p}{1-e^2}$, which is here negative, since $e > 1$.

$$\therefore \quad BO = \frac{p}{1-e^2}.$$

Now $$a = \frac{pe}{e^2-1}.$$

$$\therefore \quad OB = \frac{a}{e}.$$

Also $$\begin{aligned} OF &= OB + p \\ &= \frac{p}{e^2-1} + p \\ &= \frac{pe^2}{e^2-1} \\ &= ae. \end{aligned}$$

Since $ae = \sqrt{a^2 + b^2}$, the focus may be obtained by using the center of the conic as a center and the hypotenuse of the right triangle whose sides are a and b as a radius and describing an arc to cut the major axis produced.

176. Summary. In an hyperbola of transverse axis $2a$, conjugate axis $2b$ and eccentricity e, the following relations hold:

$$\boldsymbol{a^2 e^2 = a^2 + b^2},$$

$$\boldsymbol{ae} = \textbf{distance from center to focus},$$

$$\boldsymbol{\frac{a}{e}} = \textbf{distance from center to directrix}.$$

Compare Art. 173.

EXERCISE XXXVII

1. Derive the equation of the parabola whose directrix is the line $x = 6$, and whose focus is $(2, 3)$.

2. Derive the equation of an ellipse whose directrix is the line $y = 4$, focus at $(0, 2)$, and center at $(0, -1)$.

3. Derive the equation of the hyperbola of eccentricity 2, with focus at $(0, 4)$ and the line $x = 2$ as directrix.

4. What is the eccentricity of the equilateral hyperbola?

5. Keeping the major axis unchanged, plot ellipses with eccentricity .1, .5, .9.

What limiting position do the foci approach as the eccentricity approaches the limit 0? What is the limiting form of the ellipse?

177. The equation of the conic in polar coördinates.

(*a*) **Origin at the focus.** Taking the origin at the focus and the initial line perpendicular to the directrix, the polar equation of the conic is easily written.

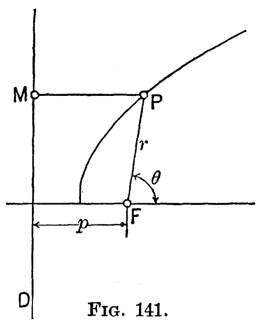


FIG. 141.

Let $P(r, \theta)$ be any point on the conic and MP the length of the perpendicular from P to the directrix. Then, by the definition of the conic,

$$FP = eMP,$$

or $$r = e(p + r\cos\theta),$$

from which

$$r = \frac{ep}{1 - e\cos\theta}.$$

If the focus lies to the left of the directrix, then

$$PM = p - r\cos\theta.$$

$$\therefore\quad r = e(p - r\cos\theta),$$

from which

$$r = \frac{ep}{1 + e\cos\theta}.$$

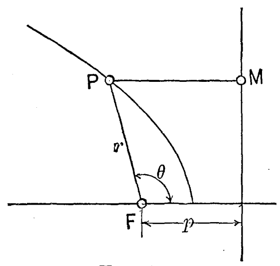


FIG. 142.

(*b*) **Origin at the center.** For the centric conics the equation in rectangular coördinates is

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

the upper sign being for the ellipse, the lower for the hyperbola.

Change to polar coördinates by means of

$$x = r \cos \theta,$$
$$y = r \sin \theta.$$

Substituting and clearing of fractions,

$$r^2b^2 \cos^2\theta \pm r^2a^2 \sin^2\theta = a^2b^2,$$

from which

$$r^2 = \frac{a^2b^2}{b^2 \cos^2\theta \pm a^2 \sin^2\theta}.$$

This equation may be expressed in a somewhat simpler form in terms of the eccentricity and b. For convenience consider separately the equation of the ellipse. It is

$$r^2 = \frac{a^2b^2}{b^2 \cos^2\theta + a^2 \sin^2\theta} = \frac{a^2b^2}{b^2 \cos^2\theta + a^2(1 - \cos^2\theta)}$$
$$= \frac{b^2}{1 - \frac{a^2 - b^2}{a^2} \cos^2\theta},$$

or

$$\boldsymbol{r^2 = \frac{b^2}{1 - e^2 \cos^2\theta}},$$

since $$e^2 = \frac{(a^2 - b^2)}{a^2}.$$

Similarly for the hyperbola the equation is

$$\boldsymbol{r^2 = \frac{-b^2}{1 - e^2 \cos^2\theta}}.$$

EXERCISE XXXVIII

Determine the nature of the following conics and sketch them:

1. $r = \dfrac{4}{1 - \frac{3}{2}\cos\theta}$.

2. $r = \dfrac{3}{2 + 4\cos\theta}$.

3. $r = \dfrac{5}{2 - 2\cos\theta}$.

4. $r^2 = \dfrac{-6}{1 - 4\cos^2\theta}$.

5. $r^2 = \dfrac{3}{2 - \cos^2\theta}$.

6. $r^2 = \dfrac{20}{2 + 3\sin^2\theta}$.

7. $r^2 = \dfrac{64}{16 - 20\sin^2\theta}$.

8. $r = a\sec^2\dfrac{\theta}{2}$.

9. Show that if the vertex of a parabola is taken as origin and the axis of the parabola as the initial line, the equation in polar coördinates is

$$r = \frac{2p\cos\theta}{\sin^2\theta}.$$

CHAPTER XIV

PROPERTIES OF CONICS

178. In this chapter a few of the more important properties of the conics are derived.

I. PROPERTIES OF THE PARABOLA

179. Subtangent of the parabola. In Art. 130 the equation of the tangent to the parabola $y^2 = 2px$ at (x_0, y_0) was found to be

$$yy_0 = p(x + x_0).$$

Letting $y = 0$ in this equation, there results $x = -x_0$,

i.e. $OT = -x_0$ (Fig. 143).

$\therefore \quad TO = x_0.$

$\therefore \quad \boldsymbol{TM = 2x_0}.$

The line TM is called the **subtangent**.

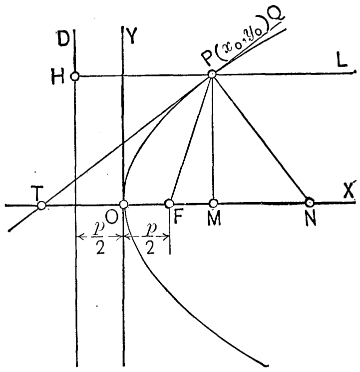


FIG. 143.

180. The subnormal of the parabola. The slope of the normal to the parabola $y^2 = 2px$ at the point (x_0, y_0) is the negative reciprocal of the slope of the tangent at that point; *i.e.* the slope of the normal is $\frac{-y_0}{p}$. The equation of the normal is therefore

$$y - y_0 = -\frac{y_0}{p}(x - x_0).$$

To find where the normal cuts the x-axis, let $y = 0$. The result is

$$x = x_0 + p.$$

I.e. in Fig. 143. $ON = x_0 + p.$

$$MN = p.$$

The line MN is called the **subnormal.**

Hence, *in the parabola the subnormal is constant and equal to p.*

181. Property of reflection of the parabola. In Fig. 143, Art. 179, from the definition of the parabola,

$$FP = HP = \frac{p}{2} + x_0.$$

Also $TF = TO + OF = x_0 + \frac{p}{2}.$ (Art. 179.)

$\therefore$ $FP = TF.$

$\therefore$ $\angle FPT = \angle FTP = \angle TPH.$

Let PL be drawn parallel to the axis of the parabola. Then

$$\angle FPT = \angle LPQ.$$

Hence, if the parabola were a reflector, any ray of light from the focus striking the parabola and reflected so as to make the angle of reflection equal to the angle of incidence would be reflected along a parallel to the axis of the parabola.

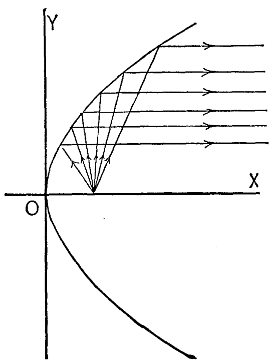


Fig. 144.

A concave reflecting surface in the form of a surface generated by revolving a parabola about its axis would therefore reflect all rays from a source at the focus in lines parallel to the axis of the reflector.

Definition. The chord of a conic which passes through the focus, perpendicular to the axis of the conic, is called the **latus rectum.**

EXERCISE XXXIX

1. By means of the result found in Art. 179, show how to draw a tangent at any point of the parabola.

2. Prove that the tangents at the ends of the latus rectum meet at the intersection of the directrix and the axis of the parabola, and are at right angles to each other.

3. Prove that the distance from the focus of a parabola to a tangent is half the length of the normal from the point of tangency to the axis of the parabola.

4. Prove that any point P of the parabola and the intersections of the axis of the parabola with tangent and normal at P are all equidistant from the focus.

5. Prove that the tangent at any point of a parabola meets the directrix and latus rectum produced at points equally distant from the focus.

6. Show that the normal at one extremity of the latus rectum of a parabola and the tangent at the other extremity are parallel.

7. Show that the directrix of a parabola is tangent to the circle described on any chord through the focus as a diameter.

8. Show that the tangent at the vertex of a parabola is tangent to the circle described on any focal radius as a diameter.

9. Prove that the angle between two tangents to a parabola is equal to one half the angle between the focal chords drawn to the points of contact.

10. Prove that the tangents at the ends of any focal chord of a parabola meet on the directrix.

11. Prove that the length of the latus rectum of the parabola $y^2 = 2px$ is $2p$.

12. Prove that if from a point (x_0, y_0) two tangents are drawn to the parabola, the equation of the line through the points of tangency is $y_0y = p(x + x_0)$.

13. By means of the preceding example prove that if tangents are drawn to the parabola from any point on the directrix, the line through the points of tangency passes through the focus.

14. Prove that in the parabola $y^2 = 2px$, the ordinate of the middle point of a chord of slope m is $\frac{p}{m}$, and hence that the locus of the middle points of a system of parallel chords of slope m is the straight line $y = \frac{p}{m}$. Draw the figure.

DEFINITION. The straight line which bisects a system of parallel chords of a parabola is called a **diameter** of the parabola.

15. Find the equation of the diameter which bisects all chords of slope m in the parabola $x^2 = 2py$. *Ans.* $x = mp$.

16. Transform the equation of the parabola $y^2 = 2px$ to the tangents at the extremities of the latus rectum as axes.

SUGGESTION. First, moving to parallel axes through $(-\frac{p}{2}, 0)$, the equation becomes

$$y^2 = 2px - p^2.$$

Next, rotating the axes through $-45°$, the equation becomes

$$x^2 - 2xy + y^2 - 2\sqrt{2}\,p(x+y) + 2p^2 = 0,$$

which becomes a perfect square on the left by the addition of $4xy$.

Then extract square root, transpose, extract square root again, and obtain

$$\sqrt{x} \pm \sqrt{y} = \pm\sqrt{p\sqrt{2}},$$

or

$$\sqrt{x} \pm \sqrt{y} = \pm\sqrt{a},$$

where

$$a = p\sqrt{2}.$$

17. Plot the curve

$$x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = \pm a^{\frac{1}{2}}.$$

What portions of the curve correspond to the different combination of signs?

II. PROPERTIES OF THE ELLIPSE AND OF THE HYPERBOLA

182. Focal radii of the ellipse. Let $P(x_0, y_0)$ be any point of the ellipse of semi-axes a and b, and let r and r' be the radii from the foci F and F' to P.

Through P draw a line parallel to the major axis of the ellipse, meeting the directrices in M and M'. Then from the definition of the ellipse, using the left-hand focus and directrix,

$$\frac{r'}{M'P} = e,$$

or

$$\boldsymbol{r' = eM'P = e\left(\frac{a}{e} + x_0\right) = a + ex_0}.$$

Similarly, using the right-hand focus and directrix,

$$r = ePM = e\left(\frac{a}{e} - x_0\right) = a - ex_0.$$

Adding, $$r + r' = 2a.$$

Hence the sum of the focal radii of a point on the ellipse is constant, and equal to the major axis of the ellipse.

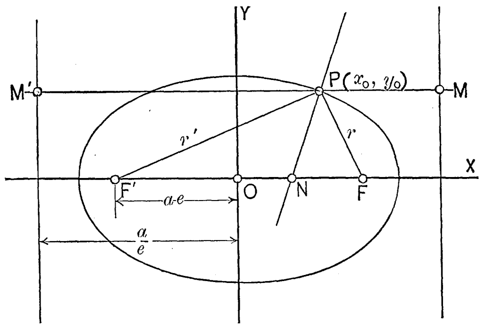


FIG. 145.

183. Focal radii of the hyperbola. In a manner similar to the above the student can show that in the hyperbola the focal radii are $r = ex_0 + a$ and $r' = ex_0 - a$, and hence

$$r - r' = 2a.$$

184. Property of reflection of the ellipse. The focal radii to any point of an ellipse make equal angles with the normal to the ellipse at that point.

PROOF. In Art. 130, the equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

at (x_0, y_0) was found to be

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

The slope of the normal at (x_0, y_0) is therefore $\frac{a^2 y_0}{b^2 x_0}$, and the

equation of the normal is

$$y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0).$$

In this equation let $y = 0$ and solve for x,

$$x = x_0 - \frac{b^2 x_0}{a^2} = \frac{a^2 - b^2}{a^2}\, x_0 = e^2 x_0.$$

I.e. in Fig. 145.

$$ON = e^2 x_0.$$

$\therefore$ $$F'N = ae + e^2 x_0 = e(a + ex_0),$$

and $$NF = ae - e^2 x_0 = e(a - ex_0).$$

$\therefore$ $$\frac{F'N}{NF} = \frac{a + ex_0}{a - ex_0} = \frac{r'}{r} \text{ (Art. 182).}$$

Therefore by plane geometry, $\angle F'PN = \angle NPF$, which proves the theorem. Hence if the ellipse served as a reflector, a ray of light, or sound, emitted at one focus would be reflected to the other.

It is on this principle that whispering galleries are sometimes constructed.

185. Property of reflection of the hyperbola. In the hyperbola the focal radii to any point of the curve make equal angles with the tangent at that point.

The proof is left to the student.

186. If a line is drawn to cut the hyperbola in two points, the two segments of the line included between the hyperbola and its asymptotes are equal.

Proof. The equations of the hyperbola, its asymptotes, and any line are, respectively,

$$b^2x^2 - a^2y^2 = a^2b^2, \tag{1}$$

$$b^2x^2 - a^2y^2 = 0, \tag{2}$$

and $$y = mx + c. \tag{3}$$

Let the points of intersection of line and hyperbola be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, and of line and asymptotes be $Q_1(x_1', y_1')$ and $Q_2(x_2', y_2')$.

Substituting the value of y from eq. (3) in eqs. (1) and (2) respectively, and collecting terms, there results

$$(b^2 - a^2m^2)x^2 - 2\,a^2mcx - a^2(c^2 + b^2) = 0 \tag{4}$$

and

$$(b - a^2m^2)x^2 - 2\,a^2mcx - a^2c^2 = 0. \tag{5}$$

The roots of (4) are x_1 and x_2, and of (5) are x_1' and x_2'. Now in any quadratic equation the sum of the roots is equal to minus the coefficient of the first power of the variable divided by the coefficient of the second power; and since the first two terms in eqs. (4) and (5) are the same, therefore

$$x_1 + x_2 = x_1' + x_2'.$$

But $\frac{x_1 + x_2}{2}$ and $\frac{x_1' + x_2'}{2}$ are respectively the abscissas of the middle points of P_1P_2 and Q_1Q_2.

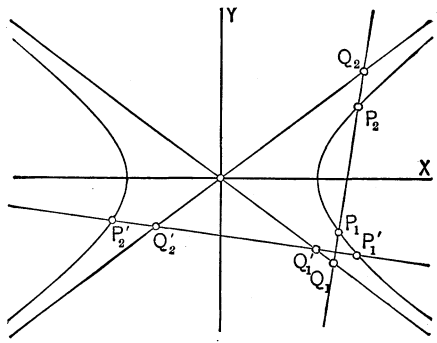


FIG. 146.

$\therefore$ the middle points of P_1P_2 and Q_1Q_2 coincide.

$\therefore$ $$Q_1P_1 = Q_2P_2.$$ Q.E.D.

EXERCISE XL

1. Prove that the length of the latus rectum of an ellipse or an hyperbola is $\frac{2b^2}{a}$.

2. Prove that the tangents at the extremities of the latus rectum of an ellipse or hyperbola intersect on the directrix.

3. Prove that the line drawn from the focus to the intersection of a tangent and the directrix of an ellipse or hyperbola is perpendicular to the line from the focus to the point of tangency.

4. A circle is drawn on the major axis of an ellipse as a diameter. A perpendicular to the major axis meets the ellipse and circle in P and Q respectively. Prove that the tangents drawn at P and Q intersect on the major axis. Hence show how to construct a tangent to an ellipse at a given point.

5. Show that the distance from the focus to an asymptote of an hyperbola is equal to b.

6. Prove that the product of the perpendiculars from any point of an hyperbola upon the asymptotes is constant, and equal to $\frac{b^2}{e^2}$.

7. Prove that the product of the perpendiculars from the foci upon a tangent to the ellipse is equal to the square of the semi-minor axis.

8. State and prove a like property of the hyperbola.

9. Prove that if tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from an exterior point (x_0, y_0), the equation of the line through the points of tangency is $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$.

10. Prove the statement in example 9 to be true for the hyperbola, with proper changes of sign.

11. Prove that if tangents are drawn to an ellipse or hyperbola from any point on the directrix, the line joining the points of tangency passes through the focus. (Use examples 9 and 10.)

12. Through a fixed point within a given circle, a circle is drawn tangent to the given circle; prove that the locus of its center is an ellipse. Draw the figure.

13. A line $y = mx + c$ cuts the ellipse $b^2x^2 + a^2y^2 = a^2b^2$; prove that if (x_1, y_1) is the middle point of the chord, then

$$x_1 = -\frac{a^2mc}{b^2 + a^2m^2}, \quad y_1 = \frac{b^2c}{b^2 + a^2m^2}.$$

14. From the preceding example, by eliminating c, prove that the locus of the middle points of a system of parallel chords, with slope m, of the ellipse is the straight line

$$y = -\frac{b^2}{ma^2}x.$$

This line is called a **diameter** of the ellipse.

Prove that any line through the center of an ellipse is a diameter.

15. Show that if two lines through the center of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

have slopes m and m' such that $mm' = -\frac{b^2}{a^2}$, then each line bisects all chords parallel to the other.

Draw two such lines.

Two such lines are called **conjugate diameters.**

16. Prove that in the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the equation of the locus of the middle points of a system of parallel chords of slope m is

$$y = \frac{b^2}{ma^2}x.$$

17. Through the point (x_0, y_0) on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ a diameter is drawn; prove that the coördinates of the extremities of its conjugate diameter are $x = \pm\frac{ay_0}{b}$, $y = \mp\frac{bx_0}{a}$.

18. If a' and b' are the lengths of two conjugate semi-diameters of the ellipse, prove that $a'^2 + b'^2 = a^2 + b^2$. (Use example 17.)

19. Prove that the tangent at any point of the ellipse is parallel to the diameter which is conjugate to the diameter through the given point; and hence that the tangents at the extremities of two conjugate diameters form a parallelogram.

20. Prove that the area of the parallelogram formed by the tangents at the extremities of two conjugate diameters of an ellipse is constant, and is equal to $4\,ab$.

Suggestion. The area in question is 8 times the area of the triangle whose vertices are $(0, 0)$, (x_0, y_0), and $\left(\frac{ay_0}{b}, -\frac{bx_0}{a}\right)$. (See example 17.)

CHAPTER XV

THE GENERAL EQUATION OF SECOND DEGREE IN TWO VARIABLES

187. In the preceding chapters certain equations of second degree in two variables have been studied. It will now be shown that every equation of second degree in two variables with real coefficients is the equation either of one of the conics, a circle, a pair of straight lines, one straight line, a point, or else the equation has no locus.

Moreover, the conditions which the coefficients must satisfy in the different cases will be established.

188. The general equation of second degree in x and y is

$$\boldsymbol{ax^2 + bxy + cy^2 + dx + ey + f = 0.} \tag{1}$$

Let the origin be moved by a translation of axes to the point (h, k) by means of the formulas

$$x = x' + h,$$
$$y = y' + k.$$

Equation (1) then becomes

$$ax'^2 + bx'y' + cy'^2 + d'x' + e'y' + f' = 0, \tag{2}$$

where
$$d' = 2\,ah + bk + d, \tag{3}$$
$$e' = bh + 2\,ck + e, \tag{4}$$
$$f' = ah^2 + bhk + ck^2 + dh + ek + f. \tag{5}$$

Equation (2) will be simplified if h and k can be so chosen that $d' = 0$ and $e' = 0$. Putting $d' = 0$ and $e' = 0$ and solving for h and k,

$$\boldsymbol{h = \frac{2\,cd - be}{b^2 - 4\,ac}}, \quad \boldsymbol{k = \frac{2\,ae - bd}{b^2 - 4\,ac}}. \tag{6}$$

These values of h and k are definite finite values unless $b^2 - 4\,ac = 0$, in which cases there are no values of h and k that make $d' = 0$ and $e' = 0$.

Hence there are two cases to consider, I, $b^2 - 4\,ac \neq 0$, and II, $b^2 - 4\,ac = 0$.

CASE I. $\boldsymbol{b^2 - 4\,ac \neq 0}$.

189. Removal of the terms of first degree. Consider first the case where $b^2 - 4\,ac \neq 0$. Then if h and k have the values shown in eq. (6), d' and e' are both zero, and eq. (2) becomes

$$ax'^2 + bx'y' + cy'^2 + f' = 0. \tag{7}$$

The value of f' can be obtained by substituting the values of h and k from (6) in (5), but more easily as follows: Multiply eq. (3) by h, eq. (4) by k, and add. The result is

$$d'h + e'k = 2\,ah^2 + 2\,bhk + 2\,ck^2 + dh + ek.$$

To both members of this equation add $dh + ek + 2f$. Then

$$d'h + e'k + dh + ek + 2f = 2(ah^2 + bhk + ck^2 + dh + ek + f) = 2f'$$

or $$2f' = dh + ek + 2f, \text{ since } d' = e' = 0.$$

Substituting the values of h and k from eq. (6),

$$f' = \frac{-(4\,acf + bde - ae^2 - cd^2 - fb^2)}{b^2 - 4\,ac}. \tag{8}$$

The quantity in the parenthesis is of importance in what follows. For convenience let it be denoted by a single letter, H;

$$\boldsymbol{H \equiv 4\,acf + bde - ae^2 - cd^2 - fb^2.}$$

Also let $$\boldsymbol{D \equiv b^2 - 4\,ac.}$$

190. Removal of the term in *xy*. Equation (7) may be reduced to one lacking the xy-term by a proper rotation of the axes.

Let
$$x' = x'' \cos\theta - y'' \sin\theta,$$
$$y' = x'' \sin\theta + y'' \cos\theta.$$

Substituting these values in eq. (7), it becomes

$$a'x''^2 + b'x''y'' + c'y''^2 + f' = 0, \tag{9}$$

where

$$a' = a \cos^2\theta + b \cos\theta \sin\theta + c \sin^2\theta, \tag{10}$$

$$b' = -2a \cos\theta \sin\theta + b(\cos^2\theta - \sin^2\theta) + 2c \cos\theta \sin\theta, \tag{11}$$

$$c' = a \sin^2\theta - b \sin\theta \cos\theta + c \cos^2\theta. \tag{12}$$

Now let θ be so chosen that $b' = 0$, *i.e.* let

$$b(\cos^2\theta - \sin^2\theta) = 2(a - c) \cos\theta \sin\theta,$$

or
$$b \cos 2\theta = (a - c) \sin 2\theta,$$

or
$$\tan 2\theta = \frac{b}{a - c}. \tag{13}$$

Since the tangent of an angle may have any real value, it is always possible to choose θ so that $b' = 0$.

With this value of θ, eq. (9) becomes

$$a'x''^2 + c'y''^2 + f' = 0. \tag{14}$$

191. Locus of the equation. The nature of the locus of eq. (14) depends upon the signs of a', c', and f', and these signs depend upon the original coefficients of eq. (1).

To determine the signs of a' and c' one may proceed as follows: Using the relations

$$2 \sin\theta \cos\theta = \sin 2\theta,$$

$$2 \cos^2\theta = 1 + \cos 2\theta,$$

$$2 \sin^2\theta = 1 - \cos 2\theta,$$

eqs. (10) and (12) may be written

$$2a' = a + c + b \sin 2\theta + (a - c) \cos 2\theta. \tag{15}$$

$$2c' = a + c - b \sin 2\theta - (a - c) \cos 2\theta. \tag{16}$$

Adding,
$$a' + c' = a + c. \tag{17}$$

Subtracting,
$$a' - c' = b \sin 2\theta + (a - c) \cos 2\theta.$$

From equation preceding (13),

$$b \cos 2\theta - (a - c) \sin 2\theta = 0.$$

Square and add

$$b \sin 2\theta + (a - c)\cos 2\theta = a' - c',$$

and

$$b \cos 2\theta - (a - c)\sin 2\theta = 0,$$

and there results

$$b^2 + (a - c)^2 = (a' - c')^2.$$

From (17)

$$(a + c)^2 = (a' + c')^2.$$

Subtracting,

$$4a'c' = 4ac - b^2. \qquad (18)$$

Since $4ac - b^2 \neq 0$, neither a' nor c' can be zero. Eq. (14) may therefore be written, since from eq. (8), $f' = -\dfrac{H}{D}$,

$$\frac{x''^2}{\dfrac{H}{a'D}} + \frac{y''^2}{\dfrac{H}{c'D}} = 1, \quad \text{if } H \neq 0, \qquad (19)$$

or

$$a'x''^2 + c'y''^2 = 0, \quad \text{if } H = 0. \qquad (20)$$

Two cases must here be considered.

(1) $\boldsymbol{D < 0}$, *i.e.* $\boldsymbol{4ac - b^2 > 0}$.

Then neither a nor c can be zero, and a and c must be of like signs. It follows also from eq. (18) that a' and c' must be of like signs, and hence of the same sign as a and c, by (17).

Therefore, if $\dfrac{H}{a} < 0$ the locus of (19) is an ellipse if $a' \neq c'$, and a circle if $a' = c'$.

If $\dfrac{H}{a} > 0$, eq. (19) has no locus.

Equation (20) is satisfied only by the point $x'' = 0$, $y'' = 0$.

(2) $\boldsymbol{D > 0}$, *i.e.* $\boldsymbol{4ac - b^2 < 0}$.

It follows from (18) that a' and c' are of opposite signs.

Equation (19) is therefore the equation of an hyperbola whether H is positive or negative.

Equation (20) can be factored, and its locus is therefore the pair of intersecting straight lines

$$\sqrt{a'}\,x'' + \sqrt{-c'}\,y'' = 0, \quad \sqrt{a'}\,x'' - \sqrt{-c'}\,y'' = 0.$$

192. Condition that eq. (1) represents a circle. It was shown in the preceding article that the locus of eq. (1) is a circle when $D < 0, \dfrac{H}{a} < 0$, and $a' = c'$. The third of these conditions can be expressed in terms of the original coefficients of eq. (1) as follows: In (17) and (18) put $a' = c'$. Then

$$2\,a' = a + c,$$

and

$$4\,a'^2 = 4\,ac - b^2.$$

Substituting in the second of these equations the value of a' from the first, there results

$$(a - c)^2 + b^2 = 0.$$

This can be satisfied by real values of a, b, and c when and only when $a = c$ and $b = 0$.

Hence the conditions that eq. (1) represents a circle are

$$D < 0,\ \frac{H}{a} < 0,\ b = 0, \text{ and } a = c.$$

CASE II. $\boldsymbol{b^2 - 4\,ac = 0}$.

193. Pass now to the case where $b^2 - 4\,ac = 0$. In this case not both a and c can be zero, for then b would be zero and eq. (1) would be only of the first degree. Moreover, a and c must be of the same sign if neither is zero. Assume at first that sign to be positive. Then eq. (1) may be written

$$ax^2 \pm 2\sqrt{ac}xy + cy^2 + dx + ey + f = 0,$$

or

$$(\sqrt{a}x \pm \sqrt{c}y)^2 + dx + ey + f = 0, \tag{21}$$

the $\pm$ sign being chosen according as b is positive or negative. In this formula $\sqrt{a}$ and $\sqrt{c}$ are real and positive.

Choose now an angle θ such that

$$\sqrt{a} = k\cos\theta,\ \pm\sqrt{c} = k\sin\theta. \tag{22}$$

Squaring and adding,

$$k = \sqrt{a + c},\ k > 0. \tag{23}$$

Then $\sqrt{a}x \pm \sqrt{c}y = k(x \cos\theta + y \sin\theta)$.

Transform now to axes which make the angle θ with the axes x and y, for which the formulas of transformation are

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta.$$

Then $x \cos\theta + y \sin\theta = x'$, and eq. (21) becomes

$$k^2x'^2 + d'x' + e'y' + f = 0, \tag{24}$$

where
$$d' = d \cos\theta + e \sin\theta, \tag{25}$$
$$e' = -d \sin\theta + e \cos\theta. \tag{26}$$

If $e' \neq 0$ eq. (24) may be written

$$y' = -\frac{k^2}{e'}x'^2 - \frac{d'}{e'}x' - \frac{f}{e'}. \tag{27}$$

This is of the form

$$y = ax^2 + bx + c$$

which in Art. 81 was seen to be the equation of a parabola.

If $e' = 0$ eq. (24) becomes

$$k^2x'^2 + d'x' + f = 0. \tag{28}$$

This is a quadratic in x' alone. It is satisfied by

$$x' = \frac{-d' \pm \sqrt{d'^2 - 4k^2f}}{2k^2}.$$

Hence eq. (28) is the equation of two parallel lines, one line, or has no locus according as

$$d'^2 - 4k^2f \gtreqless 0.$$

194. Evaluation of e' and of $d'^2 - 4k^2f$. The quantities e' and $d'^2 - 4k^2f$ of the preceding article may be expressed in terms of the coefficients of eq. (1) as follows: From (26) and (22)

$$e' = -d\left(\pm\frac{\sqrt{c}}{k}\right) + \frac{e\sqrt{a}}{k} = \frac{1}{k}(e\sqrt{a} \mp d\sqrt{c}). \tag{29}$$

Now since $D=0$; $b^2=4\,ac$, and H becomes

$$\begin{aligned} H &= bde - ae^2 - cd^2 \\ &= \pm 2\sqrt{ac}\cdot de - ae^2 - cd^2 \\ &= -(e\sqrt{a} \mp d\sqrt{c})^2, \end{aligned}$$

or, from (29), $H=-k^2e'^2$. (30)

$\therefore$ e' vanishes or does not vanish according as H does or does not vanish.

Again, from (25) and (22)

$$d' = \frac{1}{k}(d\sqrt{a} \pm e\sqrt{c}),$$

and hence

$$\begin{aligned} d'^2 - 4\,k^2f &= \frac{1}{k^2}(d\sqrt{a} \pm e\sqrt{c})^2 - 4\,k^2f \\ &= \frac{1}{k^2}[ad^2 \pm 2\,de\sqrt{ac} + e^2c - 4f(a+c)^2]. \end{aligned}$$

But if $e'=0$, then $e\sqrt{a} = \pm d\sqrt{c}$, from eq. (29).

$$\begin{aligned} \therefore \quad d'^2 - 4\,k^2f &= \frac{1}{k^2}\left[ad^2 + 2\,d^2c + \frac{d^2c^2}{a} - 4f(a+c)^2\right] \\ &= \frac{(a+c)^2}{k^2}\cdot\frac{d^2 - 4\,af}{a} \\ &= \frac{a+c}{a}(d^2 - 4\,af). \end{aligned}$$

Hence the sign of $d'^2 - 4\,k^2f$ is the same as the sign of $d^2 - 4\,af$. Therefore if $D=0$ and a is positive, the locus of (1) is a parabola if $H \neq 0$, and is two parallel lines, one line, or there is no locus according as

$$d^2 - 4\,af \gtreqless 0, \text{ if } H=0.$$

If a is negative, eq. (1) may be divided by -1 and then the above conditions hold if each coefficient in H and $d^2 - 4\,af$ is changed in sign. This, however, only changes the sign of H and does not affect at all $d^2 - 4\,af$. The above conditions hold, therefore, whether a is positive or negative.

If $a=0$, then $b=0$, and $c \neq 0$. Eq. (1) then reduces to

$$cy^2+dx+ey+f=0, \tag{31}$$

or

$$dx=-cy^2-ey-f.$$

This is a parabola if $d \neq 0$.

When $a=b=0$, H becomes $-cd^2$, and since $c \neq 0$, H vanishes or does not vanish according as d does or does not vanish.

If $H=0$ eq. (31) becomes

$$cy^2+ey+f=0,$$

which is satisfied by

$$y=\frac{-e\pm\sqrt{e^2-4\,cf}}{2\,c},$$

the locus of which is two parallel lines, one line, or there is no locus according as

$$e^2-4\,cf \gtreqless 0.$$

195. Summary. The nature of the locus of the general equation of the second degree

$$ax^2+bxy+cy^2+dx+ey+f=0$$

is shown in the following table, in which

$$D \equiv b^2-4\,ac,$$
$$H \equiv 4\,acf+bde-ae^2-cd^2-fb^2.$$

$$D<0\left\{\begin{array}{l} aH<0 \text{ ellipse, reducing to a circle if } b=0 \text{ and } a=c, \\ aH>0 \text{ no locus,} \\ H=0 \text{ a point.}\end{array}\right.$$

$$D>0\left\{\begin{array}{l} H\neq 0 \text{ hyperbola,} \\ H=0 \text{ two intersecting straight lines,}\end{array}\right.$$

$$D=0\left\{\begin{array}{l} H\neq 0 \text{ parabola,} \\ H=0\left\{\begin{array}{l} a\neq 0 \text{ two parallel lines, one line, or no locus} \\ \qquad \text{according as } d^2-4\,af \gtreqless 0, \\ a=0 \text{ two parallel lines, one line, or no locus} \\ \qquad \text{according as } e^2-4\,cf \gtreqless 0.\end{array}\right.\end{array}\right.$$

EXERCISE XLI

Apply the above test to determine the nature of the loci of the following equations.

1. $x^2 - 2\,xy + 3\,y^2 + 2\,x - y + 3 = 0.$
2. $3\,x^2 - 4\,xy + y^2 - x + 2\,y - 1 = 0.$
3. $3\,x^2 + 5\,xy - 2\,y^2 - 3\,x + y = 0.$
4. $9\,x^2 - 6\,xy + y^2 - 3\,x + y - 2 = 0.$
5. $x^2 - 4\,xy + 4\,y^2 + 2\,x - 4\,y + 1 = 0.$
6. $x^2 - xy + y^2 + 2\,x + y + 2 = 0.$
7. $3\,x^2 - 3\,xy + 3\,y^2 + 6\,x + 3\,y + 7 = 0.$
8. $4\,x^2 - 4\,xy + y^2 + 4\,x - 2\,y + 2 = 0.$
9. $4\,x^2 - 12\,xy + 9\,y^2 + x - y + 1 = 0.$
10. $3\,x^2 - xy - y^2 + x - 2\,y + 1 = 0.$

11. Show that the locus of $2\,x^2 - 2\,xy + y^2 - 3\,x + y + f = 0$ is an ellipse, a point, or there is no locus, according as f is less than, equal to, or greater than $\frac{5}{4}$.

12. Show that the locus of $ax^2 + bxy + cy^2 = 0$ is two intersecting lines, one line, or a point, according as $b^2 - 4\,ac$ is greater than, equal to, or less than zero.

13. Show that the locus of $xy + dx + ey + f = 0$ is an hyperbola except when $f = de$. What is the locus then?

14. In the equation $(lx + my + n)^2 + px + qy + r = 0$ show that $D = 0$, and that $H = -(mp - lq)^2$, and hence that the locus of the equation is a parabola except when $\frac{l}{p} = \frac{m}{q}$. What is the locus then?

CHAPTER XVI

EMPIRICAL EQUATIONS

196. Statement of the problem. It is sometimes desirable to find an equation of a curve drawn through points determined by pairs of corresponding values of two variable quantities. Frequently these values are found by experiment, and the general law which they satisfy may be known or suspected. The following illustrations will show how, in some of the simpler cases, the law may be tested and the constants of the equation determined.

The more difficult problems of this nature can be treated by the use of Fourier's series, a method of wide application, but too difficult to discuss here.

197. Points lying on a straight line. The simplest case that occurs is that where the points whose coördinates are the two measured quantities lie on, or approximately on, a straight line. In this case one has only to select the straight line which seems to best fit the points, and write its equation. The equation of this line is then the equation connecting the variables if the same scale has been used throughout. In plotting the points, however, any convenient scales may be used, and the equation of the line written with any other scale that is desired. The two coördinates in the equation of the line must then be expressed in terms of the two variables between which an equation is sought. The substitution of these values in the equation of the line gives the desired equation.

Example. The extension of a certain wire when loaded

was observed to be as shown in the following table, where E is the elongation in inches, and W is the load in pounds.

W	1	2	3	4	5	7	10	12	15	18
E	.12	.23	.34	.46	.58	.80	1.16	1.39	1.74	2.09

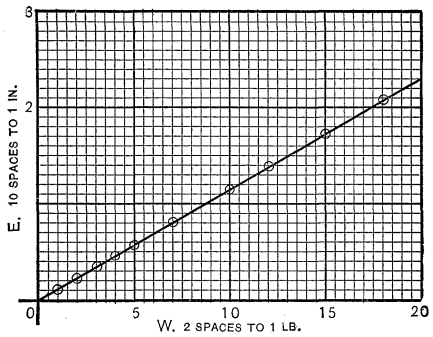


FIG. 147.

On plotting the points whose coördinates are the corresponding values of E and W, they are seen to lie approximately on a straight line. The line which seems to best fit the points passes through the origin and the point (38, 22). The equation of this line is therefore $y = \frac{22}{38}x$. But in the scale used, $x = 2\,W$, $y = 10\,E$, and hence the equation connecting E and W is $10\,E = \frac{22}{19}W$, or

$$E = .116\,W.$$

This equation therefore holds approximately for the particular wire used and within the limits of the observed values.

EXERCISE. From the following corresponding values of u and v determine the equation connecting them.

u	1	1.5	2.3	3.1	3.8	4.2	5.0	5.8	6.5	7.2	8.0
v	5.5	6.4	8.2	9.7	11.0	11.9	13.5	15.0	16.5	18.0	19.5

198. The curve $y = Cx^n$. A number of curves obtained from physical measurement follow the law $y = Cx^n$.

If the logarithm of both sides of the equation be taken, there results

$$\log y = \log C + n \log x.$$

If now $u = \log x$, $v = \log y$, $b = \log C$, then

$$v = b + nu.$$

This is an equation of first degree in u and v. Hence if u and v be taken as coördinates and the points representing corresponding values plotted, these points will lie on a straight line.

Conversely, if the points $(\log x, \log y)$ do lie on a straight line, the equation of the line is of the form

$$v = nu + b, \text{ where } u = \log x,\ v = \log y;$$

i.e. $\log y = n \log x + \log C$, if $b = \log C$,

or $\log y = \log (Cx^n)$.

$\therefore$ $y = Cx^n$.

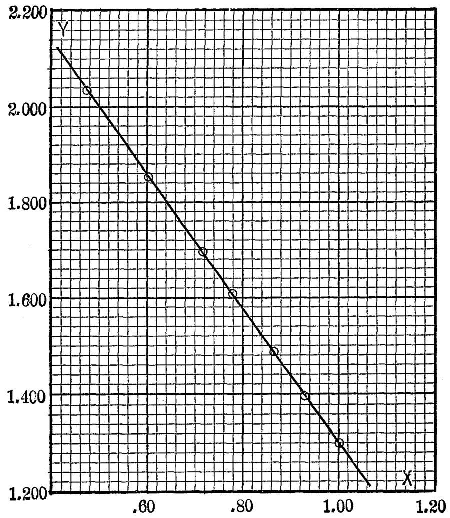


FIG. 148.

The following illustration will show how the constants C and n may be determined when the points lie on a curve of this kind.

ILLUSTRATION. The following represent pressure p and volume v of a gas:

v	3	4	5.2	6.0	7.3	8.5	10
p	107.3	71.5	49.5	40.5	30.8	24.9	19.8

Let $x = \log v$, $y = \log p$. Then the values of x and y are

x	.477	.602	.716	.778	.863	.929	1.000
y	2.031	1.854	1.695	1.607	1.489	1.396	1.297

The points determined by x and y are seen to lie on a straight line, approximately, Fig. 148. The slope of this line is found by measurement to be $-\frac{94}{67}$, or -1.40. Then, since the line passes through (1, 1.297), its equation is

$$y - 1.297 = -1.40\,(x - 1),$$

or

$$y = -1.40\,x + 2.697.$$

But

$$2.697 = \log 497.7.$$

Therefore, since

$$y = \log p, \quad x = \log v,$$

$$\log p + 1.40 \log v = \log 497.7,$$

or

$$pv^{1.40} = 497.7,$$

which is therefore approximately the formula connecting p and v.

The correctness of this formula should be tested by substituting some or all of the values of p and v in the given table.

E.g. if

$$v = 5.2 \text{ and } p = 49.5,$$

then

$$\log v = .7160,$$

which multiplied by 1.40 gives

$$\begin{aligned} \log 5.2^{1.40} &= 1.0024 \\ \log 49.5 &= \underline{1.6946} \\ \log pv^{1.40} &= 2.6970 \\ pv^{1.40} &= 497.7, \end{aligned}$$

which checks the result already found.

Where the points do not lie so accurately on the line as in this example, it would be better after obtaining the slope of the line to write $pv^n = C$, and having n, substitute the given values of p and v to find C. Make this computation for each pair of values given, and take the average of the values found for C.

Exercise. Find the equation connecting Q and h from the following observed values.

h	.583	.667	.750	.834	.876	.958
Q	7.00	7.60	7.94	8.42	8.68	9.04

199. The curve $y = ab^x$, or $y = ae^{kx}$, where $e = 2.71828 \cdots$. Certain physical quantities are connected by an equation of the form $y = ab^x$ where a and b are constant. If it is thought that two quantities for which several corresponding values are known obey this law, they may be tested, and, if the law is fulfilled, the values of the constants determined as follows: Plot the points whose abscissas are x and whose ordinates are $\log y$. If they lie on a straight line, the supposed equation is correct, otherwise not. This follows from the fact that if

$$y = ab^x, \tag{1}$$

then

$$\log y = \log a + x \log b, \tag{2}$$

and the converse.

Suppose the points $(x, \log y)$ lie on a straight line. The slope of this line is then the value of $\log b$ (see eq. (2)), from which b may be found. Also the intercept of the line on the axis of ordinates is $\log a$. From this intercept a may then be found. However, it will be more accurate to obtain a from the average of the values of $\frac{y}{b^x}$ after b is determined from the line.

In some cases a is 1, and this will be indicated by the straight line passing through the origin.

If it is desired to express $y = ab^x$ in the form $y = ae^{kx}$, one

has only to let $b = e^k$, for then $b^x = (e^k)^x = e^{kx}$. To determine k,

$$\log b = k \log e, \text{ or } k = \frac{\log b}{\log e} = \frac{\log b}{.4343}.$$

EXAMPLE. The values of x and y of the following table are thought to be connected by an equation of the form $y = ab^x$.

x	2	3.2	4.7	8.5	10.3	12.6
y	7.086	12.64	26.07	163.0	388.4	1178

Form then the following table:

x	2	3.2	4.7	8.5	10.3	12.6
$\log y$	.8504	1.1017	1.4161	2.2122	2.5893	3.0711

Plot the points $(x, \log y)$. They are seen to lie on a straight line.

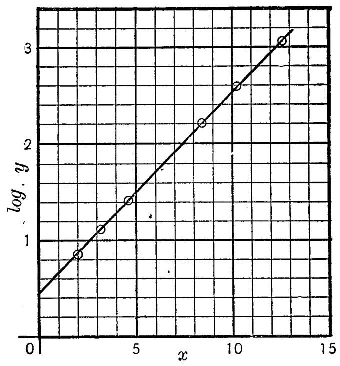


FIG. 149.

The slope of this line, computed by using the extreme values of x and $\log y$ in the table, is

$$\frac{3.0711 - .8504}{12.6 - 2} = \frac{2.2207}{10.6} = .2095.$$

Hence
$$\log b = .2095,$$
$$b = 1.62.$$

To determine a,
$$a = \frac{y}{b^x}.$$
$$\log a = \log y - x \log b$$
$$= \log y - .2095\, x.$$

Using x and $\log y$ from the table of values, the following values of $\log a$ are obtained.

.4314, .4313, .4314, .4314, .4314, .4314.

The average of these is
$$\log a = .4314.$$
$$\therefore a = 2.70.$$
$$\therefore y = 2.70(1.62^x).$$

200. Some special substitutions. In some other cases, if the law connecting the variables is suspected, the correctness of the supposition may be easily tested by a substitution which will reduce the problem to that of the straight line.

For example, if it is thought that the relation is $y = a + \frac{b}{x^2}$, plot the points $\left(\frac{1}{x^2}, y\right)$. If these points lie on a straight line, the assumed equation is correct, and the quantities a and b can be found from the graph.

In like manner the equation $xy = ax + by$, an hyperbola, may be written
$$y = a + b\frac{y}{x}, \tag{1}$$
or
$$x = b + a\frac{x}{y}, \tag{2}$$
or
$$1 = \frac{a}{x} + \frac{b}{y}, \tag{3}$$
and these may be reduced to the straight line form by using u for $\frac{y}{x}$ in (1), u for $\frac{x}{y}$ in (2), and u and v for $\frac{1}{x}$ and $\frac{1}{y}$ in (3).

Exercise. Prove that the following points lie on a curve of the form $xy = ax + by$, and determine a and b.

x	1.59	1.96	2.27	3.12	5.00	7.15	16.7
y	.885	1.11	1.28	1.85	3.24	10.8	65.0

201. The curve $y = a + bx + cx^2 + dx^3 + \cdots + kx^n$. When no other equation can be found to fit the given points the equation

$$y = a + bx + cx^2 + dx^3 + \cdots + kx^n$$

may be assumed, and by substituting the coördinates of the given points enough equations can be obtained for the determination of the constants $a, b, c, \cdots k$.

The number of terms to assume will depend upon the number and location of the given points. If the curve on which the points lie diverges only slightly and in one direction from a straight line, it will usually be sufficient to assume three terms on the right. This, of course, makes the curve a parabola. But each case must be settled on its merits, and the construction of the curve from the equation which is found will be the test of the accuracy with which it fits the given points.

Example. To find the equation of a curve through the following points:

x	8	23	39	53	63
y	10	19	27	33	36

These points when plotted are seen to lie on a curve which resembles a portion of a parabola with axis parallel to the y-axis. It is worth while then to try

$$y = a + bx + cx^2.$$

Take the two extreme points and the middle point for the determination of the coefficients. The equations obtained are

$$\begin{aligned} 10 &= a + 8\,b + 64\,c, \\ 27 &= a + 39\,b + 1521\,c, \\ 36 &= a + 63\,b + 3969\,c. \end{aligned}$$

Solving these equations,

$$a = 4.63, \qquad b = .697, \qquad c = -.00315.$$

Hence the approximate equation is

$$y = 4.63 + .697\,x - .00315\,x^2.$$

The substitution of the intermediate values of x not used in the computation of a, b, and c give,

$$\text{for } x = 23, \qquad y = 18.99,$$
$$\text{for } x = 53, \qquad y = 32.72,$$

which are reasonably close to the values of 19 and 33.

If greater accuracy is desired, four or five terms may be assumed on the right and then four or five of the given points used to determine the constants.

Again, different sets of the given points might be used to determine the constants and average values of the constants so found used.

EXERCISE XLII

1. In an experiment to determine the deflection of a beam of varying length the following measurements were made:

Length (in.)	12	16	20	24	28	32	36	40
Deflection (in.)	.17	.043	.085	.145	.220	.342	.512	.713

Prove that the deflection d and the length L are connected by an equation of the form

$$d = CL^n,$$

and find the values of n and C.

2. Find an equation connecting x and y to fit the following values:

x	6	1.2	1.6	2.2	2.8	3.4	4.3	6.0
y	.801	1.70	2.54	3.98	5.58	7.32	10.17	16.22

3. Prove that the following values of x and y satisfy an equation of the form

$$y = \frac{ax}{1 + bx},$$

and find the values of a and b.

x	.5	1.2	2.0	3.4	4.1	5.3
y	1.08	2.41	3.71	5.56	6.33	7.46

4. The following numbers are taken from a table:

x	1.1	1.4	2.0	2.6	3.4	4.1	6.3	7.8	9.8
y	.095	.336	.693	.956	1.224	1.386	1.841	2.054	2.282

Find the equation connecting x and y.

SUGGESTION. Plot the points $(\log x, y)$.

5. Prove that the following values of u and v satisfy an equation of the form $v = a + \frac{b}{u^2}$, and find the values of a and b:

u	.5	1.1	1.7	2.3	5.1	6.4
v	13.6	4.00	2.37	1.84	1.33	1.28

6. Find an equation to fit the following values of p and v:

(Try $pv^n = C$.)

v	4.2	4.7	5	5.5	6.2	7	8	9
p	105	92	86	78	68	60	53	46

ANALYTIC GEOMETRY OF SPACE

CHAPTER XVII

COÖRDINATES IN SPACE

202. Rectangular coördinates in space. As on a straight line one quantity was required to determine the position of a point, and in the plane two quantities, so in space three quantities are necessary. One way of choosing these quantities is the following: Through any point O, chosen as an origin, draw three mutually perpendicular lines OX, OY, OZ. These lines determine three mutually perpendicular planes XY, XZ, YZ. From any point P in space let perpendiculars be drawn to the three planes. Then the distances measured from the planes to the point are called the **rectangular coördinates** of the point P.

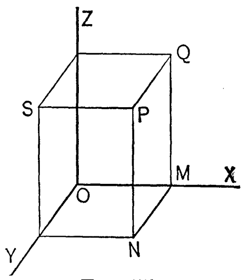


FIG. 150.

Let distances measured in the direction of OX, OY, and OZ, *i.e.* to the right, forward, and upward, be counted as positive, and distances in the opposite direction, *i.e.* to the left, backward, and downward, as negative. Then to every set of three real numbers there corresponds a point in space and conversely.

The distances SP, QP, and NP (Fig. 150) are called respectively the $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ of the point P, and the point is denoted by (x, y, z), or by $P(x, y, z)$.

The plane containing OX and OY is called the **$\boldsymbol{xy}$-plane**, and similarly for the others.

The three planes containing the axes are known as **coördinate planes.**

The eight portions of space separated by the coördinate planes are called **octants.**

Two points are said to be **symmetric** with respect to a plane when the line joining the points is perpendicular to the plane and is bisected by it.

EXERCISE XLIII

1. Locate the points (1, 3, 2), (− 1, 3, 4), (1, − 2, 4), (1, 3, − 2), (2, − 3, − 4), (− 1, − 2, − 3), (− 1, − 2, 3), (− 1, 3, − 2), (0, 1, 2), (2, 0, 0), (0, 0, 0).

2. Show that the line OP in Fig. 150 is the diagonal of a rectangular parallelopiped of which the numerical values of x, y, and z are the lengths of the sides.

3. Show that $OP = \sqrt{x^2 + y^2 + z^2}$.

4. Find the distance from the origin to each of the points, (1, 3, − 2), (3, − 1, 4), (2, − 1, − 3).

5. Find the point symmetric to each of the following points with respect to each of the coördinate planes, (2, 3, 4), (− 3, − 1, − 2), (3, − 1, 2).

6. Find the point symmetric to each of the following points with respect to the origin, (2, 3, 5), (− 2, 4, 3), (3, − 4, − 1).

7. Prove that (a, b, c) and $(-a, -b, -c)$ are symmetric with respect to the origin.

8. What is the value of x for any point in the yz-plane? What therefore is the equation of the yz-plane? What are the equations of the other coördinate planes?

9. Where do all points lie that have $x = 0$ and $y = 0$? What are the equations of the coördinate axes?

10. Find the locus of points which satisfy the following sets of conditions:

(a) $x = y,\ z = 0$.

(b) $x = y,\ z = 2$.

(c) $y = z,\ x = -2$.

(d) $x = y = z$.

(e) $-x = y,\ y = z$.

(f) $x = 2,\ y = 3$.

(g) $x^2 + y^2 = 16,\ z = 0$.

(h) $\dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} = 1,\ x = 0$.

(i) $y^2 = 4x,\ z = 3$.

203. Distance between two points in rectangular coördinates. Let the points be $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Through P_1 and P_2 pass planes parallel respectively to the three coördinate planes. These three planes form a rectangular parallelopiped of which P_1P_2 is the diagonal, and the edges are respectively the differences of the coördinates parallel to the edges.

Thus, in Fig. 151, $P_1N = x_2 - x_1$, $NM = y_2 - y_1$, $MP_2 = z_2 - z_1$.

But $$\overline{P_1P_2}^2 = \overline{P_1N}^2 + \overline{NM}^2 + \overline{MP}_2^2.$$

$$\therefore \boldsymbol{d = P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}.$$

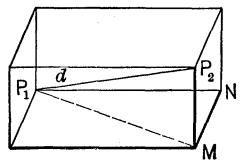


FIG. 151.

If the two points are the origin and the point (x, y, z), this formula becomes

$$\boldsymbol{d = \sqrt{x^2 + y^2 + z^2}}.$$

204. Point dividing a line in a given ratio. If the point (x, y, z) divides the line from (x_1, y_1, z_1) to (x_2, y_2, z_2) in the ratio $r : 1$, then

$$\boldsymbol{x = \frac{x_1 + rx_2}{1 + r}, \quad y = \frac{y_1 + ry_2}{1 + r}, \quad z = \frac{z_1 + rz_2}{1 + r}}.$$

The proof is left to the student.

EXERCISE XLIV

1. Find the distance between $(3, 4, -2)$ and $(-5, 1, -6)$.

2. Prove that the center of gravity, *i.e.* the intersection of the medians of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

3. Show that the lines drawn from the vertices of a tetrahedron to the intersection of the medians of the opposite faces meet in a common point which is $\frac{3}{4}$ the distance from each vertex to the opposite face.

4. Write the equation which expresses the condition that (x, y, z) shall be equidistant from $(0, 0, 0)$ and $(3, 5, 1)$. What is the locus of (x, y, z)?

5. Write the condition that (x, y, z) shall remain at the distance 4 from $(0, 0, 0)$. What is the locus of (x, y, z)?

6. Find the equation of the surface of a sphere with center at $(2, 1, -3)$ and radius 5.

205. Polar coördinates. A point in space may be determined by its distance from the origin and the angles which the line from the origin to the point makes with the rectangular coördinate axes.

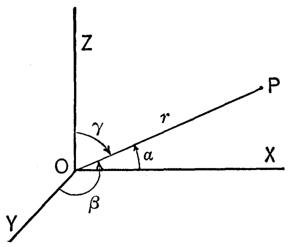


FIG. 152.

Thus, let OX, OY, OZ, be a set of rectangular axes, and let P be any point in space. Then OP and the angles α, β, γ, between OP and the axes of x, y, and z, respectively, determine the position of P. If $OP = r$, the point may be denoted by $(r, \alpha, \beta, \gamma)$. The four quantities r, α, β, γ are sometimes called the **polar coördinates** of P.

It is convenient to restrict r, α, β, γ to positive values, and to further restrict the angles to values not greater than 180°. Any point in space may be represented by such values of r, α, β, γ.

The angles α, β, γ are called the **direction angles** of OP, and the cosines of these angles the **direction cosines** of OP.

206. Relation between rectangular and polar coördinates of a point. From Fig. 153, if the rectangular coördinates of P are x, y, z, then the following relations are seen to hold:

$$\mathbf{x = r\cos\alpha,}$$
$$\mathbf{y = r\cos\beta,}$$
$$\mathbf{z = r\cos\gamma.}$$

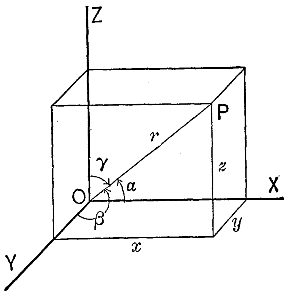


FIG. 153.

Since $r = \sqrt{x^2+y^2+z^2}$, the above equations may be solved for the direction cosines and the following values obtained:

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}},$$
$$\cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}},$$
$$\cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

207. Relation between the direction cosines of a line.

DEFINITION. The direction cosines of a given directed line are the direction cosines of a line drawn from the origin in the same direction as the given line.

If the three equations of the preceding article,

$$x = r\cos\alpha,$$
$$y = r\cos\beta,$$
$$z = r\cos\gamma,$$

be squared and added, there is obtained

$$x^2+y^2+z^2 = r^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma).$$

But $$x^2+y^2+z^2 = r^2.$$

$$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1.$$

Hence, *the sum of the squares of the direction cosines of any straight line is* 1.

208. Direction cosines of a line joining two points. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two points in space and consider the line as directed from P to P_2.

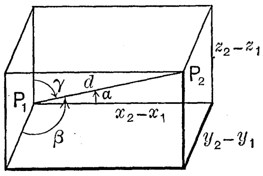


FIG. 154.

Let the direction cosines of P_1P_2 be $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Then $$\cos\alpha = \frac{x_2 - x_1}{d}, \quad \cos\beta = \frac{y_2 - y_1}{d}, \quad \cos\gamma = \frac{z_2 - z_1}{d},$$

where $$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

These relations are evident from Fig. 154.

209. Spherical coördinates. Again, take the three mutually perpendicular axes OX, OY, OZ.

Let P be any point in space. Then the position of P is determined by the distance r, or OP, and the angles θ and ϕ, where θ is the angle between OP and the positive OZ, and ϕ is the angle between the positive OX and the orthogonal projection of OP upon the xy-plane.

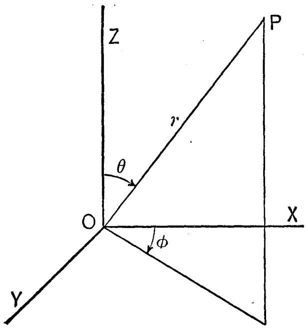


FIG. 155.

The point is denoted by (r, θ, ϕ).

The quantities r, θ, and ϕ are called the **spherical coördinates** of P.

The student can easily show

that if P has rectangular coördinates (x, y, z), then the relations between the rectangular and spherical coördinates of the point are

$$x = r \sin \theta \cos \phi,$$
$$y = r \sin \theta \sin \phi,$$
$$z = r \cos \theta.$$

Spherical coördinates are useful in some surveying and astronomical problems.

EXERCISE XLV

1. Find the direction cosines of the line from the origin to $(2, -1, 3)$.

2. Show that if any three real quantities, a, b, c, be chosen, a line with direction cosines proportional to these quantities can be found, and that the direction cosines are $\frac{a}{d}, \frac{b}{d}, \frac{c}{d}$, where $d = \sqrt{a^2 + b^2 + c^2}$.

3. Find the direction cosines of the line from $(3, 1, -2)$ to $(-1, 4, 3)$. Draw the figure.

4. Given $\cos \alpha = \frac{1}{2}$, $\cos \beta = \frac{1}{4}$, find $\cos \gamma$.

5. Find the rectangular coördinates of a point whose polar coördinates are $(2, 30°, 45°, \gamma)$. How many solutions?

6. Find the spherical coördinates of a point whose rectangular coördinates are $(3, 2, 4)$.

7. Find the spherical coördinates of a point in terms of the rectangular coördinates of the point.

8. Show that reversing the direction of a line changes the sign of each direction cosine.

9. Write the direction cosines of each coördinate axis.

210. Projection of a line upon another line.

Definition. From the extremities A and B of a line AB drop perpendiculars upon a line MN, meeting it in C and D respectively. Then CD is called the **orthogonal projection** of AB upon MN. (Fig. 156.)

Only orthogonal projection will be used in what follows,

and projection will be understood to mean orthogonal projection.

Definition. The angle between two non-intersecting lines is defined to be the angle between two intersecting lines drawn in the same directions respectively as the given lines.

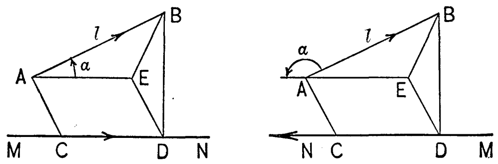


Fig. 156.

If α is the angle between AB and MN, and l is the length of AB, then

$$l \cos \alpha = \text{projection of } AB \text{ on } MN.$$

Proof. Through B pass a plane perpendicular to MN and through A draw a line parallel to MN to cut this plane in E. (Fig. 156.)

Then $$AE = CD.$$

But $$AE = l \cos \alpha.$$

$$\therefore CD = l \cos \alpha.$$

(If $\alpha > 90°$, CD is negative, *i.e.* is opposite in direction to MN.)

211. Projection of a broken line. The projection on any axis of a straight line joining two points is equal to the sum of the projections on the same axis of the sides of any broken line connecting the two points, if the parts of the broken line are directed so that the beginning of each side after the first is at the end of the preceding.

This is evident from the definition of projection.

Thus in Fig. 157,

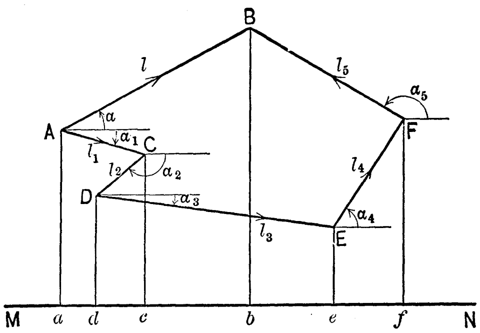


FIG. 157.

$$ab = ac + cd + de + ef + fb,$$

or $\text{proj. } AB = \text{proj. } AC + \text{proj. } CD + \text{proj. } DE + \text{proj. } EF + \text{proj. } FB.$

If l, l_1, l_2, $\cdots$ l_5 are the lengths of AB, AC, CD, $\cdots$ FB, respectively, and α, α_1, α_2, $\cdots$ α_5 are the angles between these lines and MN, then

$$l \cos \alpha = l_1 \cos \alpha_1 + l_2 \cos \alpha_2 + \cdots + l_5 \cos \alpha_5.$$

212. The angle between two lines in terms of their direction cosines. Let two lines have direction angles α_1, β_1, γ_1, and α_2, β_2, γ_2, respectively, and let θ be the angle between them. To find the value of θ.

Through the origin draw two lines OP_1 and OP_2 having the same directions respectively as the two given lines.

Let the coördinates of P_1 be (x_1, y_1, z_1) and let $OP_1 = r_1$.

On OP_2 project OP_1 and the broken line $OM + MN + NP_1$. (Fig. 158). Since

$$\text{proj. } OP_1 = \text{proj. } OM + \text{proj. } MN + \text{proj. } NP_1,$$

therefore,

$$r_1 \cos \theta = x_1 \cos \alpha_2 + y_1 \cos \beta_2 + z_1 \cos \gamma_2.$$

But $$x_1 = r_1 \cos \alpha_1,\ y_1 = r_1 \cos \beta_1,\ z_1 = r_1 \cos \gamma_1.$$

$$\therefore r_1 \cos \theta = r_1 \cos \alpha_1 \cos \alpha_2 + r_1 \cos \beta_1 \cos \beta_2 + r_1 \cos \gamma_1 \cos \gamma_2,$$

or $$\boldsymbol{\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.}$$

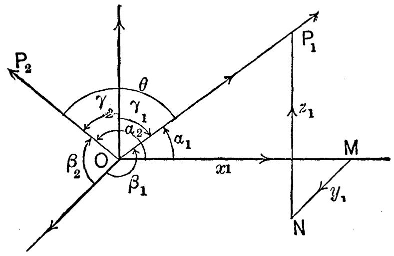


Fig. 158.

(Notice that if one, or more, of the coördinates x_1, y_1, z_1 is negative, *e.g.* y_1, then $-y_1$ is the *length* of MN, but $180° - \beta_2$ is the angle between MN and OP_2; hence the middle term is $-y_1 \cos(180° - \beta_2)$, which is the same as $y_1 \cos \beta_2$.)

EXERCISE XLVI

1. Find the projection of the line from $(2, 1, -3)$ to $(3, -4, 5)$ upon each of the coördinate axes.

2. The direction cosines of a line are proportional to 2, 3, and -4. Find their values.

3. Express in terms of the direction cosines of two lines the condition that the two lines be parallel. The condition that they be perpendicular.

4. Find the angle between two lines whose direction cosines are respectively proportional to $2, -1, 3$ and $1, 3, -2$.

CHAPTER XVIII

LOCI AND THEIR EQUATIONS

213. Certain straight lines and planes. The student has already considered some simple equations of straight lines and planes. For example, $x = a$ is the equation of a plane parallel to the yz-plane.

The two equations $y = b$, $z = c$, represent a straight line parallel to the x-axis, the intersection of the two planes $y = b$ and $z = c$.

The two equations $x = y$, $z = c$, represent a straight line, the intersection of the plane $z = c$ and a plane bisecting the dihedral angle between the xz-plane and the yz-plane.

214. Cylinders with elements parallel to a coördinate axis. Consider a circular cylinder with the z-axis for its axis and with radius r. (Fig. 159.)

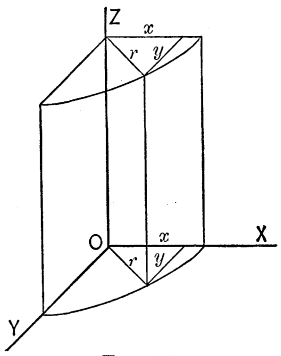


FIG. 159.

If any point P be taken on the surface of this cylinder, the x and y of the point are the same as the x and y of the projection of the point on the xy-plane. But these latter values satisfy the equation of the circle $x^2 + y^2 = r^2$. Hence the coördinates of P satisfy the same equation.

The equation of the surface of the cylinder is therefore

$$x^2 + y^2 = r^2.$$

In like manner it may be shown that if a straight line, kept always parallel to the z-axis, is moved along any curve in the xy-plane, a cylindrical surface is

generated which has the same equation as the equation of the curve in the xy-plane.

Thus the equation $y^2 = 4x$, interpreted as an equation of a locus in space, is the equation of a cylindrical surface generated by a straight line parallel to the z-axis, moving along the curve $y^2 = 4x$ in the xy-plane.

Likewise an equation of the form $y = f(z)$, read "y equals f of z," *i.e.* y is a function of z, is the equation of a cylindrical surface generated by moving a line parallel to the x-axis along the curve $y = f(z)$ in the yz-plane.

The student should describe the locus in space of the equation $z = f(x)$.

EXERCISE XLVII

Describe and sketch the loci in space of the following equations:

1. $x^2 + z^2 = 25$.
2. $(x - a)^2 + (y - b)^2 = r^2$.
3. $x \cos \alpha + y \sin \alpha = p$.
4. $\dfrac{x}{a} + \dfrac{y}{b} = 1$.
5. $x^2 = 2pz$.
6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
7. $y^2 - z^2 = a^2$.
8. $y = mz + c$.

215. Surfaces of revolution. If the equation of a curve in one of the coördinate planes is known, the equation of the surface formed by revolving this curve about one of the coördinate axes can be obtained from it.

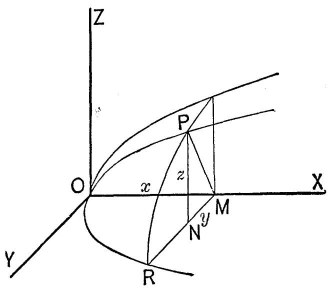


Fig. 160.

As an illustration, consider the surface formed by revolving about the x-axis the parabola $y^2 = 4x$.

Let $P(x, y, z)$ be any point on this surface. Then (Fig. 160)

$$x = OM,\ y = MN,\ z = NP.$$

Since $MP = MR$, it follows from the equation of the parabola

that

$$\overline{MP}^2 = 4\,OM.$$

But $$\overline{MP}^2 = \overline{MN}^2 + \overline{NP}^2 = y^2 + z^2.$$

$$y^2 + z^2 = 4\,x.$$

This is therefore the equation which is true for any point on the surface, and clearly for no other points, and hence is the equation of the surface.

The process of obtaining the equation of the surface from that of the curve in the xy-plane consists in replacing y by $\sqrt{y^2+z^2}$.

In general, if any curve in the xy-plane, $F(x, y) = 0$, be revolved about the x-axis, the equation of the surface formed is

$$F(x, \sqrt{y^2+z^2}) = 0.$$

EXERCISE XLVIII

1. Find the equation of the surface generated by revolving the curve $y^2 = 4\,x$ about the y-axis. Sketch the figure in one octant.

2. Find the equation of the surface generated by revolving the circle $x^2 + y^2 = r^2$ about the x-axis; about the y-axis.

3. Find the equation of the surface of the spheroid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the y-axis. The spheroid is said to be oblate if $a > b$, prolate if $a < b$.

4. Find the equation of the surface of a cone generated by revolving the line $y = mx$ about the x-axis.

216. Nature of locus determined by plane sections. It is frequently useful, in trying to determine the nature of a locus, to find the intersection of the locus by a plane. Generally the planes parallel to the coördinate axes, or else containing a coördinate axis, are the simplest ones to use.

Example 1. As an illustration, consider the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

If in this equation z be set equal to zero, the resulting equation represents the part of the locus which lies in the plane $z = 0$, *i.e.* in the xy-plane.

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the equation of the intersection of the locus of eq. (1) and the xy-plane.

This intersection is called the trace of eq. (1) in the xy-plane. It is an ellipse with semi-axes a and b lying on the axes of x and y, and with center at the origin.

Likewise the equations of the locus in the xz- and the yz-planes are shown to be respectively the ellipses

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,$$

and

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

To find the trace of the locus of eq. (1) in a plane parallel to the yz-plane let x be held constant in eq. (1) and y and z be allowed to vary. Letting $x = k$ in eq. (1), the resulting equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2},$$

in which the constant term $\frac{k^2}{a^2}$ is transposed to the right side of the equation.

This equation may be written

$$\frac{y^2}{\frac{b^2(a^2 - k^2)}{a^2}} + \frac{z^2}{\frac{c^2(a^2 - k^2)}{a^2}} = 1.$$

This is the equation of an ellipse, if $k^2 < a^2$, with axes in the planes of xy and xz, the values of the semi-axes being

$$b' = \frac{b\sqrt{a^2 - k^2}}{a} \text{ and } c' = \frac{c\sqrt{a^2 - k^2}}{a}.$$

Hence any section of the locus of eq. (1) by a plane parallel to the yz-plane, and distant less than a from the origin is an ellipse with axes in the planes of xy and xz. As k changes gradually from 0 to a, the semi-axes of the ellipse change gradually from b and c to 0. The locus of eq. (1) may then be thought of as generated by an ellipse of gradually varying dimensions moving with its axes in the planes of xy and xz. The locus is therefore a *surface*.

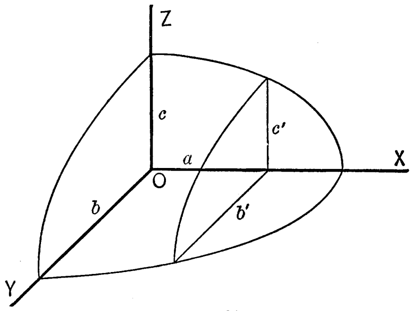


FIG. 161.

Since all sections parallel to three mutually perpendicular planes are ellipses, the figure is called an **ellipsoid** (Fig. 161.)

EXAMPLE 2. To find the locus of

$$x^2 + 2y^2 = 4z. \tag{2}$$

If z is held constant, $z = k$, the equation may be written

$$\frac{x^2}{4k} + \frac{y^2}{2k} = 1,$$

which is the equation of an ellipse if $k > 0$, but has no locus if $k < 0$. When $k = 0$, the equation is satisfied only by the point (0, 0). Therefore a section of the locus of eq. (2) by a plane parallel to the xy-plane is an ellipse if the plane is above the xy-plane, but there are no points below the xy-plane which satisfy the equation.

If $x = 0$, eq. (2) reduces to $y^2 = 2z$, which is the equation of a parabola in the yz-plane.

If $y = 0$, eq. (2) reduces to $x^2 = 4z$, which is the equation of a parabola in the xz-plane.

The locus of eq. (2) may therefore be thought of as a surface

generated by an ellipse, moving in a plane parallel to the xy-plane, its center on the z-axis, and so changing in size that the ends of its axes are always on the curves $y^2 = 2z$ and $x^2 = 4z$. (Fig. 162.)

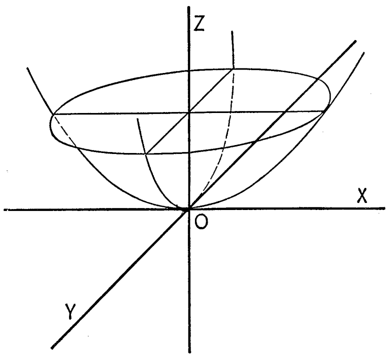


FIG. 162.

The figure is called an **elliptic paraboloid**.

217. Locus of an equation in three variables. In general an equation in three variables represents a surface. For if any one of the variables be held constant, an equation between the other two variables is obtained, which in general represents a curve, as was found in the study of loci in two variables. The locus of the equation in three variables is then such that in general its intersections by planes parallel to the coördinate planes are curves. Therefore the locus of the equation is in general a surface.

EXERCISE XLIX

Discuss and sketch the loci of the following equations:

1. $x^2 + y^2 + z^2 = r^2$.
2. $y^2 = x + z$.
3. $x + y + z = 1$.
4. $x^2 + y^2 + 4z^2 = 1$.
5. $x^2 + y^2 - z^2 = 0$.
6. $x^2 + 4z^2 = y^2$.
7. $x + y = \sin z$.

CHAPTER XIX

THE PLANE AND THE STRAIGHT LINE

I. THE PLANE

218. The normal equation of the plane. Let p be the length of the perpendicular from the origin upon a plane, and let the direction angles of this perpendicular be α, β, γ.

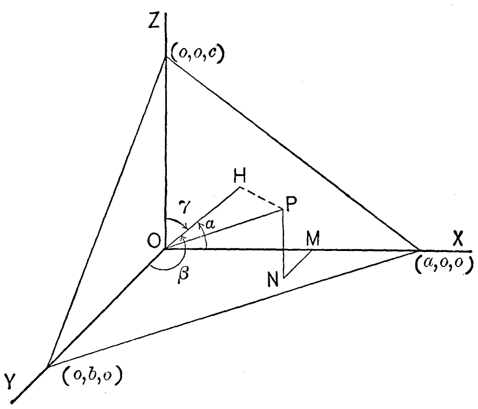


FIG. 163.

Let P (x, y, z) be any point in the plane. Project the line OP and also the broken line $OM + MN + NP$ upon the perpendicular. (Fig. 163.) These projections are equal. (Art. 211.)

$\therefore$ $x \cos\alpha + y\cos\beta + z\cos\gamma =$ proj. of OP on OH, (Art. 211)

or $\quad \boldsymbol{x\cos\alpha + y\cos\beta + z\cos\gamma = p}.$

Since this is true for any point in the plane, and for no other points, it is the equation of the plane.

It is known as the **normal equation** of the plane.

219. The intercept equation of the plane. If the above plane meets the axes in $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, then $\cos\alpha = \frac{p}{a}$, $\cos\beta = \frac{p}{b}$, $\cos\gamma = \frac{p}{c}$.

Substitute these values of $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in the equation of the plane and there results

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

220. The general equation of the first degree in x, y, and z. The general equation of first degree in x, y, and z is

$$Ax + By + Cz + D = 0. \qquad (1)$$

Consider the point Q whose coördinates are the coefficients of x, y, and z; *i.e.* the point (A, B, C). (Fig. 164.) Let OQ have direction angles α, β, γ. Then

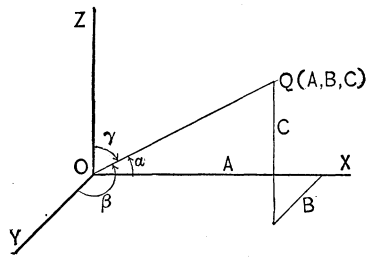


Fig. 164.

$$\cos\alpha = \frac{A}{OQ}, \cos\beta = \frac{B}{OQ}, \cos\gamma = \frac{C}{OQ},$$

where $\qquad OQ = \sqrt{A^2 + B^2 + C^2}.$

Dividing eq. (1) through by $\pm\sqrt{A^2 + B^2 + C^2}$, it may be

written in the form

$$\frac{A}{\pm\sqrt{A^2+B^2+C^2}}x+\frac{B}{\pm\sqrt{A^2+B^2+C^2}}y+\frac{C}{\pm\sqrt{A^2+B^2+C^2}}z$$
$$=\frac{-D}{\pm\sqrt{A^2+B^2+C^2}}. \qquad (2)$$

Let the sign of the radical be chosen so that $\dfrac{-D}{\pm\sqrt{A^2+B^2+C^2}}$ is positive, and let $p=\dfrac{-D}{\pm\sqrt{A^2+B^2+C^2}}$. Eq. (2) may then be written

$$x\cos\alpha'+y\cos\beta'+z\cos\gamma'=p,$$

in which α', β', γ', are the same as α, β, γ, or are the direction angles of the line from the origin to $(-A, -B, -C)$, according as the positive or negative sign of the radical is chosen.

In either case the equation is the equation of a plane by Art. 218. Therefore the equation

$$Ax+By+Cz+D=0,$$

in which A, B, C, D, are real quantities, is the equation of a plane. If p is the length of the perpendicular from the origin to the plane, and α, β, γ, are the direction angles of this perpendicular, then

$$\cos\boldsymbol{\alpha}=\frac{\boldsymbol{A}}{\pm\sqrt{\boldsymbol{A}^2+\boldsymbol{B}^2+\boldsymbol{C}^2}},\quad \cos\boldsymbol{\beta}=\frac{\boldsymbol{B}}{\pm\sqrt{\boldsymbol{A}^2+\boldsymbol{B}^2+\boldsymbol{C}^2}},$$

$\cos\boldsymbol{\gamma}=\dfrac{\boldsymbol{C}}{\pm\sqrt{\boldsymbol{A}^2+\boldsymbol{B}^2+\boldsymbol{C}^2}}$, $\boldsymbol{p}=\dfrac{-\boldsymbol{D}}{\pm\sqrt{\boldsymbol{A}^2+\boldsymbol{B}^2+\boldsymbol{C}^2}}$, the same sign of the radical being used throughout, and so chosen that p is positive.

221. Distance from a point to a plane.

(The case where the point and the origin are on opposite sides of the plane is the only one discussed here.)

Let d be the distance from (x_1, y_1, z_1) to the plane whose equation is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

Through (x_1, y_1, z_1) draw a plane parallel to the given plane. Since the perpendicular from the origin to this plane is in the same direction as that from the origin to the given plane, the equation of the second plane is

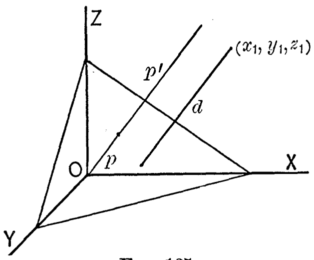


FIG. 165.

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p'.$$

Since (x_1, y_1, z_1) is on this plane,

$$x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma = p'.$$

But $\qquad d = p' - p.$

$$\therefore\ \boldsymbol{d = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p}.$$

The student should show that if the point and the origin are on the same side of the plane, the above formula gives the negative of the distance from the point to the plane.

From the above it follows that the distance from (x_1, y_1, z_1) to the plane $\qquad Ax + By + Cz + D = 0$

is
$$\boldsymbol{d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}}.$$

222. The angle between two planes. Since the angle between two planes is equal to the angle between the normals to the planes, it follows that the angle between the two planes

$$x \cos \alpha_1 + y \cos \beta_1 + z \cos \gamma_1 = p_1,$$

and
$$x \cos \alpha_2 + y \cos \beta_2 + z \cos \gamma_2 = p_2$$

is given by

$$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2,$$

and that the angle between the two planes

$$A_1x + B_1y + C_1z + D_1 = 0,$$

and
$$A_2x + B_2y + C_2z + D_2 = 0$$

is given by

$$\cos\theta = \pm \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

EXERCISE L

Find the lengths and direction cosines of the perpendiculars from the origin upon each of the following planes. Reduce each equation to the normal form.

1. $2x - 3y + 4z = 5$.
2. $3x - 5y - 2z = 0$.
3. $3x + 4y = 2$.
4. $x + y + z = 1$.

Find the distance from the following points to the planes:

5. From (3, 1, 2) to $2x - 3y + 7z = 2$.
6. From $(-1, 3, 2)$ to $x + 2y - z = 5$.
7. From (0, 0, 1) to $2x - y = 4$.
8. Find the angle between the two planes of example 1 and example 2.
9. Find the angle between the two planes of example 3 and example 4.

II. THE STRAIGHT LINE

223. The equations of a straight line through two points. Let the two given points be $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Let $P(x, y, z)$ be any point on the line through P_1 and P_2. Project P_1P and P_1P_2 upon the x-axis. Then, by plane geometry,

$$\frac{M_1M}{M_1M_2} = \frac{P_1P}{P_1P_2}.$$

$$\therefore \frac{x - x_1}{x_2 - x_1} = \frac{P_1P}{P_1P_2}.$$

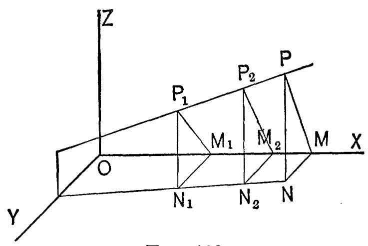


FIG. 166.

In like manner it is shown that

$$\frac{y - y_1}{y_2 - y_1} = \frac{P_1P}{P_1P_2},$$

and

$$\frac{z - z_1}{z_2 - z_1} = \frac{P_1P}{P_1P_2}.$$

$$\therefore \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

These equations are therefore the equations of the straight line.

224. The equations of a straight line through a given point and with given direction cosines. In the preceding article if the line makes angles α, β, γ with the axes, and if $d = P_1P_2$, then

$$\cos\alpha = \frac{x_2 - x_1}{d}, \quad \cos\beta = \frac{y_2 - y_1}{d}, \quad \cos\gamma = \frac{z_2 - z_1}{d}.$$

Substituting the values of $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ obtained from these equations in the equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}, \tag{1}$$

there results, on dividing through by d,

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \tag{2}$$

Hence these are the equations of a straight line through (x_1, y_1, z_1) with direction angles α, β, γ.

Any equations of the form

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

are the equations of a straight line through (x_1, y_1, z_1) with direction cosines proportional to l, m, n. For these equations have only to be multiplied by $\sqrt{l^2 + m^2 + n^2}$ to bring them into the form

$$\frac{x - x_1}{\dfrac{l}{\sqrt{l^2 + m^2 + n^2}}} = \frac{y - y_1}{\dfrac{m}{\sqrt{l^2 + m^2 + n^2}}} = \frac{z - z_1}{\dfrac{n}{\sqrt{l^2 + m^2 + n^2}}},$$

which are the same as eqs. (2), since the denominators in these equations are the direction cosines of a straight line. (Art. 206.)

225. The general equations of a straight line. Since a straight line is the intersection of two planes, the equations

of two planes may be taken as the general equations of a straight line. Thus

$$A_1x + B_1y + C_1z + D_1 = 0,$$

and

$$A_2x + B_2y + C_2z + D_2 = 0,$$

are the equations of a straight line.

Since one straight line is the intersection of an indefinite number of pairs of planes, the same straight line may correspond to an indefinite number of pairs of equations of first degree.

A line not perpendicular to the x-axis may be represented by equations of the form

$$y = mx + b,$$

and $z = nx + c$. (Fig. 167.)

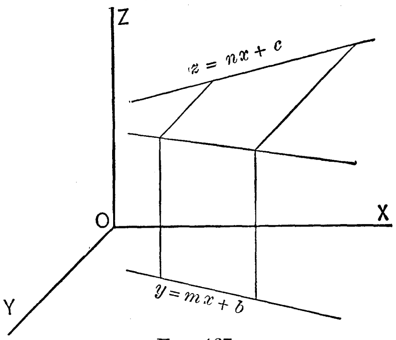


Fig. 167.

If it is perpendicular to the x-axis, but not to the y-axis, its equations may be written

$$x = a,$$
$$z = my + b. \quad \text{(Fig. 168.)}$$

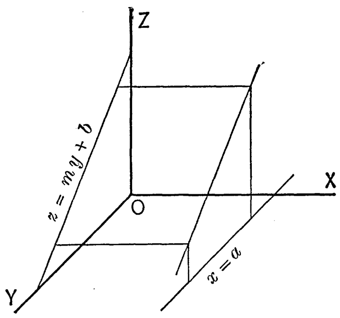


Fig. 168.

If it is perpendicular to both the x- and y-axes, *i.e.* is parallel to the z-axis, its equations may be written

$$x = a,$$
$$y = b.$$

EXERCISE LI

1. Find the direction cosines of the line

$$\frac{x-1}{5} = \frac{y-3}{2} = \frac{z+1}{3}.$$

2. Find the direction cosines of the line

$$y = 3x + 5,$$
$$z = 2x + 1.$$

3. Prove that the direction cosines of the line

$$y = mx + b,$$
$$z = nx + c$$

are proportional to 1, m, n.

4. Prove that the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

is perpendicular to the plane

$$lx + my + nz + p = 0.$$

5. Find the angle between the line

$$\frac{x-3}{2} = \frac{y}{4} = \frac{z-1}{-1}$$

and the perpendicular to the plane

$$3x - 2y + 4z = 0.$$

6. Find the angle between the lines

$$x = \frac{y-2}{2} = \frac{z-3}{3}$$

and

$$\frac{x+1}{3} = \frac{y-3}{-2} = z + 2.$$

7. Find the angle between the lines

$$3x - 2y = 4,$$
$$4y - 2z = 1;$$

and

$$x = 2y + 3 = 4z - 1.$$

8. Find the equations of the line through $(1, -1, 2)$ which makes equal angles with the axes.

9. Find the equations of a line through $(3, 4, 1)$ and $(-2, 1, 3)$.

10. Find the equations of a line through $(3, 1, -2)$ perpendicular to the plane $2x - 3y + 4z = 0$.

11. Find the equation of a plane through $(2, 1, 3)$ parallel to the line $x = 2y + 4 = 3z - 1$. Also the equation of a plane perpendicular to the given line and passing through the given point.

CHAPTER XX

THE QUADRIC SURFACES

226. Definition. The **quadric surfaces**, or **conicoids**, are surfaces whose equations are of the second degree in rectangular coördinates of space.

Certain standard forms of equations of second degree, formed by analogy to the standard equations of second degree in two variables, will be studied in the succeeding articles.

227. The ellipsoid. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

This equation has already been discussed in Art. 216. Only the figure is shown here.

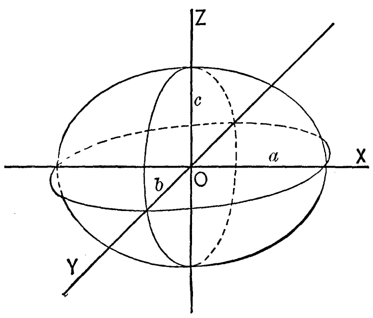


Fig. 169.

If two of the quantities, a, b, c, are equal, *e.g.* if $b=c$, the equation reduces to that of the **spheroid**,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1,\quad \textbf{prolate if } b<a,\quad \textbf{oblate if } b>a.$$

If $a = b = c$, the equation reduces to that of the **sphere**

$$x^2 + y^2 + z^2 = a^2.$$

228. The hyperboloid of one sheet.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Sections of the surface represented by this equation by planes parallel to the xy-plane are of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2},$$

or

$$\frac{x^2}{\frac{a^2(c^2+z^2)}{c^2}} + \frac{y^2}{\frac{b^2(c^2+z^2)}{c^2}} = 1.$$

If z is held constant, this is the equation of an ellipse.

Sections parallel to the xz-plane are of the form

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = \frac{b^2 - y^2}{b^2},$$

or

$$\frac{x^2}{\frac{a^2(b^2-y^2)}{b^2}} - \frac{z^2}{\frac{c^2(b^2-y^2)}{b^2}} = 1, \text{ if } b \neq y.$$

If y is held constant, this is the equation of an hyperbola with major axis parallel to the x-axis if $y^2 < b^2$, and with major axis parallel to the z-axis if $y^2 > b^2$.

For $y = b$, or $y = -b$, the equation represents two intersecting straight lines

$$\frac{x}{a} + \frac{z}{c} = 0, \text{ and } \frac{x}{a} - \frac{z}{c} = 0.$$

The hyperboloid of one sheet is sketched in Fig. 170, and a few sections parallel to the xz-plane are indicated.

Sections parallel to the yz-plane are also hyperbolas, and have their major axes parallel to the y- or z-axis according as

the distance of the section from the origin is less than or greater than a. The two sections parallel to the yz-plane at the distance a from the origin are each the pair of straight lines

$$\frac{y}{b}+\frac{z}{c}=0, \text{ and } \frac{y}{b}-\frac{z}{c}=0.$$

229. The hyperboloid of two sheets.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

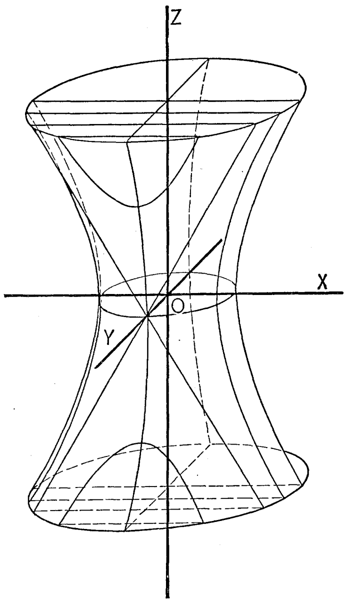


FIG. 170.

Any section of the surface parallel to the xy-plane is an hyperbola with major axes parallel to the x-axis, the major axis and conjugate axis both increasing as the distance of the cutting plane from the xy-plane increases, but their ratio remaining equal to $\frac{a}{b}$.

A like remark applies to sections of the surface made by planes parallel to the xz-plane, the major axis being parallel to the x-axis and the ratio of the axes being equal to $\frac{a}{c}$.

Sections of the surface parallel to the yz-plane are of the form

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x^2}{a^2}-1,$$

or

$$\frac{y^2}{\frac{b^2(x^2-a^2)}{a^2}}+\frac{z^2}{\frac{c^2(x^2-a^2)}{a^2}}=1.$$

This is the equation of an ellipse if $x^2>a^2$, but there is no

locus if $x^2 < a^2$. When $x = \pm a$, the locus is a point $(a, 0, 0)$, or $(-a, 0, 0)$.

Since this hyperboloid consists of two separate parts, it is called the hyperboloid of two sheets. Only one part is shown in the figure. The other part is symmetric to the part that is shown with respect to the yz-plane. (Fig. 171.)

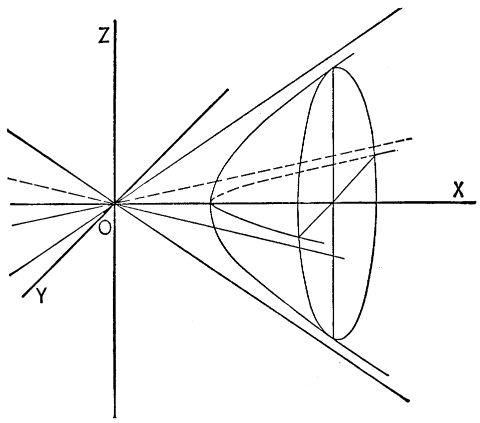


FIG. 171.

230. The elliptic paraboloid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}.$$

The trace in the xy-plane is the origin $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$. The trace in the xz-plane is the parabola $x^2 = \frac{a^2}{c} z$. The trace in the yz-plane is the parabola $y^2 = \frac{b^2}{c} z$. Sections of the surface parallel to the xy-plane are of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k}{c},$$

and are therefore ellipses if k and c are of the same sign, but there is no locus if k and c are of opposite sign. Sections of

the surface parallel to the xz-plane are the parabolas

$$\frac{x^2}{a^2}=\frac{z}{c}-\frac{k^2}{b^2}$$

with vertices at $\left[0,\ k,\ \frac{ck^2}{b^2}\right]$, and axes parallel to the z-axis. Sections of the surface parallel to the yz-plane are the parabolas

$$\frac{y^2}{b^2}=\frac{z}{c}-\frac{k^2}{a^2}$$

with vertices at $\left[k,\ 0,\ \frac{ck^2}{a^2}\right]$, and axes parallel to the z-axis.

The locus is sketched in Fig. 172, for c positive.

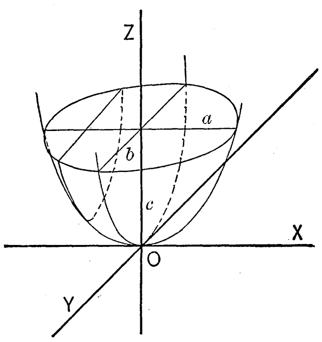


FIG. 172.

231. The hyperbolic paraboloid.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}.$$

The trace in the xy-plane is the pair of straight lines

$$\frac{x}{a}-\frac{y}{b}=0, \text{ and } \frac{x}{a}+\frac{y}{b}=0.$$

The trace in the xz-plane is the parabola $x^2=\frac{a^2}{c}z$. The trace in the yz-plane is the parabola $y^2=-\frac{b^2}{c}z$. Sections parallel to

the xy-plane are the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{k}{c}.$$

Sections parallel to the xz-plane are the parabolas

$$\frac{x^2}{a^2}=\frac{z}{c}+\frac{k^2}{b^2}.$$

Sections parallel to the yz-plane are the parabolas

$$\frac{y^2}{b^2}=-\frac{z}{c}+\frac{k^2}{a^2}.$$

The locus is sketched in Fig. 173 for c positive.

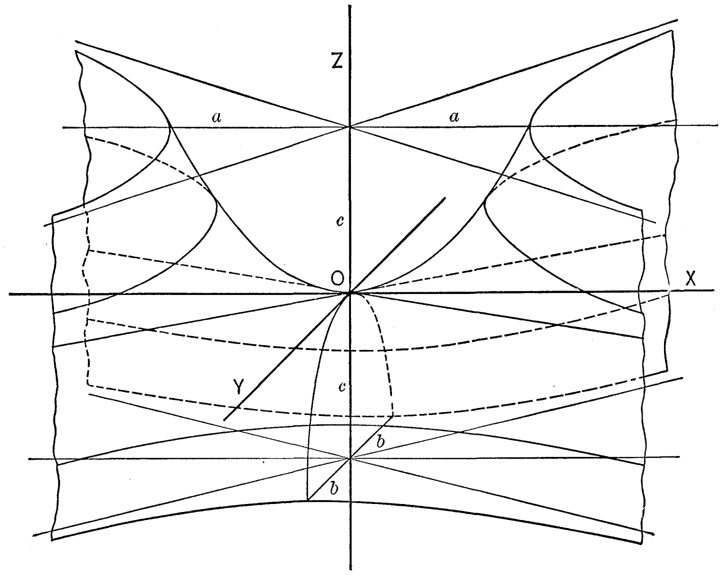


FIG. 173.

232. The cone.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

Let this surface be cut by the plane $y = x \tan \theta$. Let x' be the distance from the z-axis to any point P on the intersection of surface and plane. Then

$$y = x' \sin \theta, \quad x = x' \cos \theta, \text{ (Fig. 174)},$$

and
$$\frac{x'^2 \cos^2 \theta}{a^2} + \frac{x'^2 \sin^2 \theta}{b^2} - \frac{z^2}{c^2} = 0,$$

or
$$\frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{a^2 b^2} x'^2 - \frac{z^2}{c^2} = 0.$$

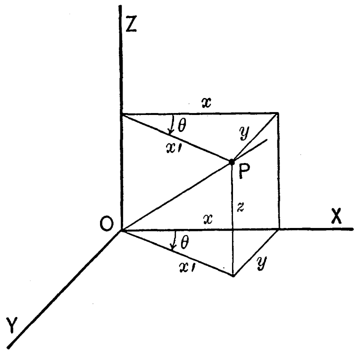


Fig. 174.

This is the equation of the intersection of the plane and surface referred to rectangular coördinates in the cutting plane. The equation can be factored into two real factors of first degree in x' and z, and is therefore the equation of two straight lines. Since $x' = 0$ and $z = 0$ reduce both of the factors to zero, the two lines pass through the origin.

Hence any plane containing the z-axis intersects the surface in two straight lines through the origin.

Moreover any plane parallel to the xy-plane, $z = k$, intersects the surface in the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2},$$

an ellipse.

Hence the locus of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

is a cone with vertex at (0, 0, 0), and with the section

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the distance c from the xy-plane. (Fig. 175.)

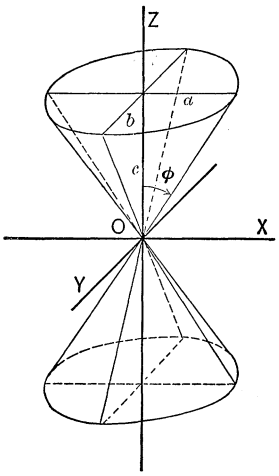


FIG. 175.

233. The right circular cone. In the equation of the preceding article if $a=b$, the cone becomes a right circular cone.

If $\frac{a}{c}$ be replaced by m, the equation of the right circular cone becomes

$$\boldsymbol{x^2+y^2-m^2z^2=0.}$$

If $x=0$, then $y=\pm mz$. Therefore the straight lines $y=\pm mz$ are the intersections of the cone and the yz-plane. Hence the quantity m is the tangent of the angle between an element of the cone and its axis. $m=\tan\phi$, in Fig. 175.

234. The conic sections. In the equation of the cone

$$x^2+y^2-m^2z^2=0,$$

let the y- and z-axes be rotated through the angle θ to the new axes OY' and OZ'. The old coördinates y and z of any point in terms of the new coördinates y' and z' of the point are then

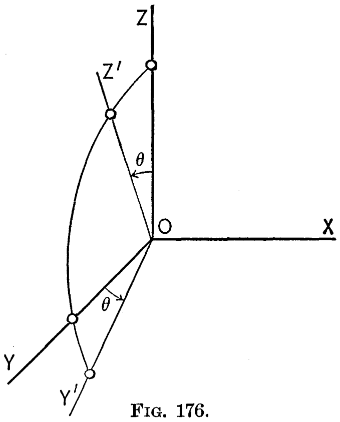


FIG. 176.

given by

$$z = z' \cos\theta - y' \sin\theta,$$
$$y = z' \sin\theta + y' \cos\theta.$$

The x-coördinate does not change. (Fig. 176.)

Substituting in the equation of the cone, and collecting terms, the equation of the cone referred to the new axes is

$$x^2 + (\sin^2\theta - m^2\cos^2\theta) z'^2 + 2\sin\theta\cos\theta(1+m^2)y'z' + (\cos^2\theta - m^2\sin^2\theta)y'^2 = 0.$$

If in this equation y' is held constant, the intersection of the cone and a plane parallel to the xz'-plane is obtained. Since the x and z' of points in this plane are the same as their projections on the xz'-plane, the equation of the curve of intersection is of the form

$$x^2 + az'^2 + dz' + f = 0,$$

where

$$a = \sin^2\theta - m^2\cos^2\theta,$$
$$d = 2\sin\theta\cos\theta(1+m^2)y',$$
$$f = (\cos^2\theta - m^2\sin^2\theta)y'^2.$$

A discussion of this equation shows that

(1) If $y' = 0$, then both d and f are zero, and the equation becomes

$$x^2 + az'^2 = 0.$$

This is the equation of a point if $a > 0$, *i.e.* if $\tan^2\theta > m^2$; of two intersecting lines if $\tan^2\theta < m^2$; and of one straight line if $\tan^2\theta = m^2$.

(2) $y' \neq 0$.

(a) If $\tan^2\theta = m^2$, the equation is of the form

$$x^2 + dz' + f = 0,$$

which is the equation of a parabola.

(b) If $\tan^2\theta \neq m^2$, the equation is of the form

$$x^2 + az'^2 + dz' + f = 0,$$

which is of the type of an ellipse or hyperbola according as a is positive or negative, *i.e.* according as $\tan^2\theta$ is greater than or less than m^2.

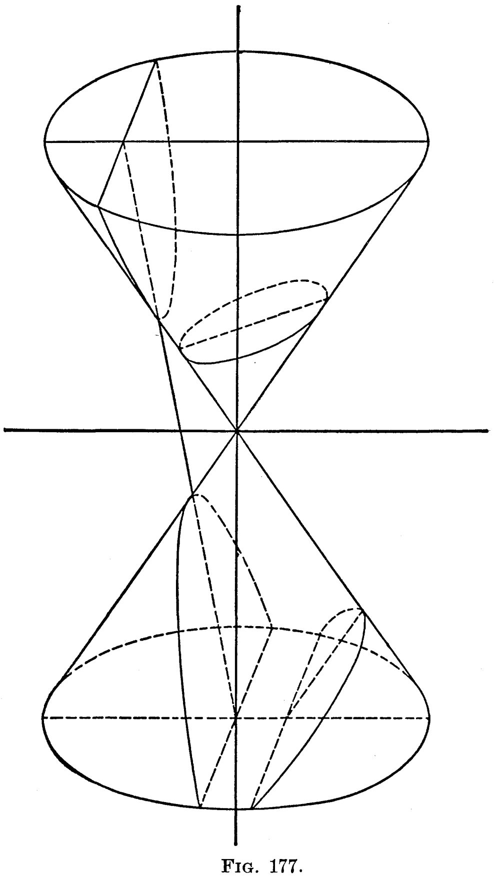


Fig. 177.

If $\theta = 90°$, *i.e.* the cutting plane is perpendicular to the axis of the cone, the equation reduces to

$$x^2 + z'^2 = m^2y'^2,$$

which is the equation of a circle.

Hence, if a right circular cone is cut by a plane:

(1) passing through the vertex, the intersection is a pair of lines, a single line, or a point, according as the angle which the plane makes with the axis of the cone is less than, equal to, or greater than the angle between the axis and an element of the cone;

(2) not passing through the vertex, the intersection is an hyperbola, a parabola, or an ellipse, according as the angle between the plane and the axis of the

cone is less than, equal to, or greater than the angle between the axis and an element of the cone.

In the special case where the plane is perpendicular to the axis of the cone, the intersection is a circle. (See Fig. 177.)

EXERCISE LII

Describe and sketch the loci of the following equations:

1. $x^2 + y^2 + 4z^2 = 4$.
2. $x^2 + y^2 - 4z^2 = 4$.

3. $z^2 + y^2 = 4x$.
4. $x^2 - 4(y^2 + z^2) = 0$.

5. $x^2 + 2z^2 = y$.
6. $z - x^2 = y^2$.

7. $\frac{y^2}{b^2} - \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$.
8. $pv = Rt$, R constant; p, v, t, variables.

9. $\frac{x^2}{a^2} + \frac{z^2}{c^2} = by$.
10. $x^2 - z^2 = 2y$.

11. $(x-1)^2 + (y+2)^2 + (z+1)^2 = 16$.

12. $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.
13. $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$.

CHAPTER XXI

SPACE CURVES

235. Introduction. In this chapter a few curves in space, which do not lie in a plane, will be considered, and the equations derived.

236. The helix. The **helix** is a curve traced on the surface of a right circular cylinder by a point which advances in the direction of the axis of the cylinder at the same time that it rotates around the axis, the amount of advance being proportional to the angle of rotation.

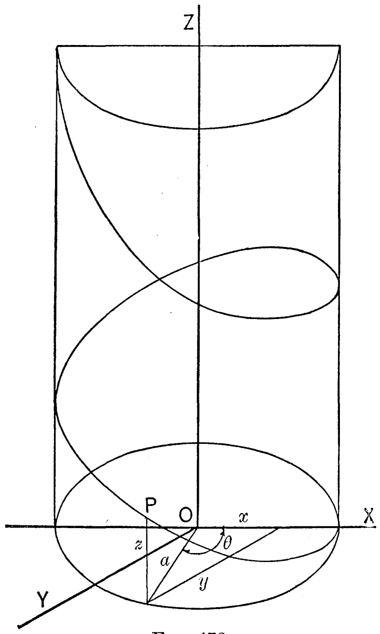


FIG. 178.

To find the equations of the helix, let the axis of z be the axis of the cylinder on which the helix is traced, a the radius of the cylinder, b the amount of advance along the axis to each radian of rotation, and let the x-axis be chosen to pass through a point of the helix. Then, if θ is the angle of rotation around the axis, the values of x, y, and z of any point on the curve are

$$x = a \cos \theta,$$
$$y = a \sin \theta,$$
$$z = b\,\theta. \qquad \text{(Fig. 178.)}$$

237. The curve of intersection of two cylinders of unequal radii, with axes intersecting at right angles.

Let the axes of the cylinders be the x- and y-axes respectively, the radii a and b respectively. (Fig. 179.) The equations of the surfaces are then

$$y^2 + z^2 = a^2,$$

and $$x^2 + z^2 = b^2.$$

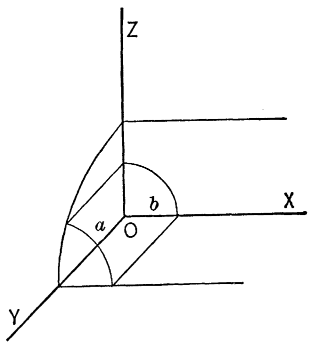


FIG. 179.

These equations, regarded as simultaneous equations, are therefore the equations of the curves of intersection.

The equations of the curve may be written in the parametric form, as in the case of the helix, by letting z equal some arbitrary function of another variable and then solving the equations for x and y. *E.g.* if

$$z = a \sin \theta,$$

then $$y = \pm\, a \cos \theta,$$

and $$x = \pm \sqrt{b^2 - a^2 \sin^2 \theta}.$$

Or z itself may be considered the parameter, and the equations written in the parametric form

$$x = \pm \sqrt{b^2 - z^2},$$
$$y = \pm \sqrt{a^2 - z^2},$$
$$z = z.$$

238. The curve of intersection of a sphere and circular cylinder.

Let the sphere have its center at the origin and radius a, and let the cylinder have its axis parallel to the x-axis, cutting the z-axis at $z = c$, and radius b.

The equation of the sphere and cylinder are then respectively

$$x^2 + y^2 + z^2 = a^2,$$

and
$$y^2 + (z - c)^2 = b^2.$$

These equations, regarded as simultaneous equations, are therefore the equations of the curve of intersection.

The student should sketch the figure.

The coördinate z may conveniently be considered the independent variable and have arbitrary values assigned to it, the corresponding values of x and y being then computed from the equations. Corresponding to one value of z four points are obtained, in general.

Exercise. Letting $a = 5$, $b = 2$, $c = 3$, find four points on the curve corresponding to $z = 2$. How many points of the curve are there having $z = 1$? How many having $z = 5$?

239. General equations of a space curve. If the equations of two surfaces are known, these equations, regarded as simultaneous equations, are satisfied by all points common to the two surfaces, and by only those points. The equations of the two surfaces are therefore together the equations of the curve of the intersection of the two surfaces.

EXERCISE LIII

1. A screw has 8 threads to the inch. The diameter of the screw is $\frac{1}{2}$ inch. What are the equations of the edge of the threads?

2. A point starts at the base of a right circular cone and traces a curve on the surface, advancing in the direction of the axis of the cone proportional to the angle of rotation about the axis. Find the equations of the curve.

3. Similar to example 2, using a hemisphere instead of a cone.

4. Find the polar equation of the projection of the curve of example 2 upon the plane of the base of the cone. Trace the curve of projection.

5. Find the polar equation of the projection of the curve of example 3 upon the plane of the base of the hemisphere.

CHAPTER XXII

TANGENT LINES AND PLANES

240. Introduction. In the plane a knowledge of derivatives was found to be important in obtaining the equations of tangent lines to curves. In space, also, derivatives play an important part in the deduction of the equations of tangent planes to surfaces and of tangent lines to curves. But in space a somewhat extended conception of derivatives is necessary, for the number of variables has increased from two to three.

241. Partial Derivatives. Consider an equation which expresses z as a function of two independent variables, x and y.

E.g.
$$z = 2\,x^2 + 3\,xy^2 + 5\,y^3. \tag{1}$$

If y is regarded as a constant and the derivative of z taken with respect to x, the result is

$$4\,x + 3\,y^2.$$

This result is called the **partial derivative** of z with respect to x, and is denoted by the symbol $\frac{\partial z}{\partial x}$. Thus,

$$\frac{\partial z}{\partial x} = 4\,x + 3y^2.$$

Similarly,
$$\frac{\partial z}{\partial y} = 6\,xy + 15\,y^2.$$

In eq. (1) let x and y take the values x_0 and y_0 respectively. Then z takes a corresponding value, z_0. Then

$$z_0 = 2\,x_0^2 + 3\,x_0y_0^2 + 5\,y_0^3.$$

Let x take an increment, Δx. Then z takes a corresponding increment. Let this increment, which is due to the change in x only, be denoted by $\Delta_x z$. Then

$$z_0 + \Delta_x z = 2(x_0 + \Delta x)^2 + 3(x_0 + \Delta x)y_0^2 + 5\,y_0^3.$$

From the definition of a derivative, it follows that the value of $\left.\frac{\partial z}{\partial x}\right|_{x=x_0}$ is the limiting value of $\frac{\Delta_x z}{\Delta x}$ as Δx approaches zero. The student can easily check this by computing the value of $\frac{\Delta_x z}{\Delta x}$ from the above equations and finding the limiting value.

In general, if $f(x, y)$, read "f of x and y," is used to denote any function of x and y, then, if

$$z = f(x, y),$$

the values of $\left.\frac{\partial z}{\partial x}\right|_{x_0, y_0}$ and $\left.\frac{\partial z}{\partial y}\right|_{x_0, y_0}$ are defined by

$$\left.\frac{\partial z}{\partial x}\right|_{x_0, y_0} = \underset{\Delta x \doteq 0}{\text{limit}} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x},$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{x_0, y_0} = \underset{\Delta y \doteq 0}{\text{limit}} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

If

$$u = F(x, y, z),$$

then $\left.\frac{\partial u}{\partial x}\right|_{x_0, y_0, z_0}$ is defined by

$$\left.\frac{\partial u}{\partial x}\right|_{x_0, y_0, z_0} = \underset{\Delta x \doteq 0}{\text{limit}} \frac{F(x_0 + \Delta x, y_0, z_0) - F(x_0, y_0, z_0)}{\Delta x},$$

and similarly for the other partial derivatives of u.

EXERCISE LIV

1. Find the partial derivative of z with respect to x and y for the values $x = 2$, $y = 3$, if $z = 3x^2 - 5xy^2 + 2y^3$.

2. In the equation of example 1, letting $x = 2$, $y = 3$, compute the increment in z due to an increment of .1 in x. Also the increment in z due to an increment of .1 in y.

3. If $u = 3x^2 + 2xyz + y^2 + 5x^2z + yz^2$, find the partial derivatives of u with respect to each of the variables x, y, z.

4. In $z = 2x^2 + 3xy + y$, find the value of $\frac{\partial z}{\partial x}$, (1) by differentiating, regarding y as constant; (2) by giving x an increment, Δx, computing $\Delta_x z$, and finding the limiting value of $\frac{\Delta_x z}{\Delta x}$.

242. The tangent plane to a surface. Let

$$F(x, y, z) = 0$$

be the equation of a surface. Let $P(x_0, y_0, z_0)$ be any point on this surface, and let the surface be cut by a plane parallel to the z-axis and passing through P. (Fig. 180.) The equation of such a plane is

$$y = mx + b.$$

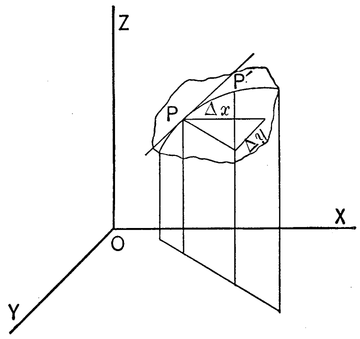


FIG. 180.

Let $P'(x_0 + \Delta x,\ y_0 + \Delta y,\ z_0 + \Delta z)$ be any other point on the intersection of the surface and plane. Then

$$y_0 + \Delta y = m(x_0 + \Delta x) + b,$$

and

$$y_0 = mx_0 + b.$$

$$\therefore \quad \Delta y = m\,\Delta x,$$

or

$$m = \frac{\Delta y}{\Delta x}.$$

The equation of the line through P and P' is

$$\frac{x - x_0}{\Delta x} = \frac{y - y_0}{\Delta y} = \frac{z - z_0}{\Delta z}, \qquad \text{(Art. 223)}$$

or

$$\frac{x - x_0}{1} = \frac{y - y_0}{\dfrac{\Delta y}{\Delta x}} = \frac{z - z_0}{\dfrac{\Delta z}{\Delta x}}.$$

Let P' approach the limiting position P. The line through P and P' approaches the limiting position of tangency at P to the curve of intersection of the plane and surface, and hence of tangency at P to the surface. The equation of this tangent line is therefore

$$\frac{x-x_0}{1}=\frac{y-y_0}{m}=\frac{z-z_0}{n} \tag{1}$$

where n is the limiting value of $\dfrac{\Delta z}{\Delta x}$ as P' approaches P.

To find the value of n, let $F(x, y, z)$ be represented by u;

$$u=F(x, y, z).$$

Since P' and P are both on the surface, therefore

$$F(x_0+\Delta x,\ y_0+\Delta y,\ z_0+\Delta z)=0,$$

and

$$F(x_0, y_0, z_0)=0.$$

$$\therefore\ \frac{F(x_0+\Delta x,\ y_0+\Delta y,\ z_0+\Delta z)-F(x_0, y_0, z_0)}{\Delta x}=0.$$

This equation may be written

$$\frac{F(x_0+\Delta x,\ y_0+\Delta y,\ z_0+\Delta z)-F(x_0,\ y_0+\Delta y,\ z_0+\Delta z)}{\Delta x}$$
$$+\frac{F(x_0,\ y_0+\Delta y,\ z_0+\Delta z)-F(x_0, y_0, z_0+\Delta z)}{\Delta y}\cdot\frac{\Delta y}{\Delta x}$$
$$+\frac{F(x_0, y_0, z_0+\Delta z)-F(x_0, y_0, z_0)}{\Delta z}\cdot\frac{\Delta z}{\Delta x}=0.$$

If Δy and Δz were held constant, and Δx allowed to approach zero as a limit, the first term of this equation would approach the limiting value

$$\left.\frac{\partial u}{\partial x}\right|_{z_0,\ y_0+\Delta y,\ z_0+\Delta z}.$$

Since, however, as P' approaches P, Δx, Δy, and Δz all approach zero, the limiting value of the first term is

$$\left.\frac{\partial u}{\partial x}\right|_{x_0, y_0, z_0}.$$

Likewise the second and third terms approach the limiting values

$$m\frac{\partial u}{\partial y}\bigg|_{x_0,\, y_0,\, z_0},$$

and

$$n\frac{\partial u}{\partial x}\bigg|_{x_0,\, y_0,\, z_0}.$$

$$\therefore \frac{\partial u}{\partial x}+m\frac{\partial u}{\partial y}+n\frac{\partial u}{\partial z}=0, \tag{2}$$

the values of the partial derivatives being taken at (x_0, y_0, z_0).

Equation (2) expresses the value of n in terms of m. If eq. (2) were solved for n, and the value substituted in eqs. (1), the equation of the tangent at P to the curve of intersection of the plane and surface would be obtained. The elimination of m between the eqs. (1) would then result in an equation between the coördinates of points on *any* tangent line to the surface at P, *i.e.* the equation of the locus of all tangent lines that can be drawn to the surface at P.

The elimination of m and n is most easily affected by solving eqs. (1) for m and n and substituting their values in eq. (2). The result is

$$(x-x_0)\frac{\partial u}{\partial x}\bigg|_{x_0,\, y_0,\, z_0}+(y-y_0)\frac{\partial u}{\partial y}\bigg|_{x_0,\, y_0,\, z_0}+(z-z_0)\frac{\partial u}{\partial z}\bigg|_{x_0,\, y_0,\, z_0}=0.$$

Since this is an equation of first degree in x, y, and z, it is the equation of a plane.

Hence all tangent lines to a surface at a given point lie in a plane. This plane is called the **tangent plane** to the surface at that point.

Since $$u=F(x, y, z),$$

the symbol $\frac{\partial F}{\partial x}$ may be used instead of $\frac{\partial u}{\partial x}$.

Hence, if $$F(x, y, z)=0$$

is the equation of any surface, then

$$(x-x_0)\frac{\partial F}{\partial x}\bigg|_{x_0,\,y_0,\,z_0}+(y-y_0)\frac{\partial F}{\partial y}\bigg|_{x_0,\,y_0,\,z_0}+(z-z_0)\frac{\partial F}{\partial z}\bigg|_{x_0,\,y_0,\,z_0}=0$$

is the equation of the tangent plane to the surface at (x_0, y_0, z_0).

243. Illustration. Consider the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Here
$$F(x, y, z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1,$$

and
$$\frac{\partial F}{\partial x}=\frac{2\,x}{a^2},\ \frac{\partial F}{\partial y}=\frac{2\,y}{b^2},\ \frac{\partial F}{\partial z}=\frac{2\,z}{c^2},$$

and hence the equation of the tangent plane at (x_0, y_0, z_0) is

$$(x-x_0)\frac{2\,x_0}{a^2}+(y-y_0)\frac{2\,y_0}{b^2}+(z-z_0)\frac{2\,z_0}{c^2}=0.$$

Since
$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1,$$

the equation of the tangent plane becomes

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1.$$

244. The normal to a surface. A line perpendicular to a tangent plane to a surface at the point of tangency is called a normal to the surface at that point.

If the equation of the surface is

$$F(x, y, z)=0,$$

the equation of the tangent plane has been found to be

$$(x-x_0)\frac{\partial F}{\partial x}\bigg|_{x_0,\,y_0,\,z_0}+(y-y_0)\frac{\partial F}{\partial y}\bigg|_{x_0,\,y_0,\,z_0}+(z-z_0)\frac{\partial F}{\partial z}\bigg|_{x_0,\,y_0,\,z_0}=0.$$

The equations of a line perpendicular to this plane and passing through (x_0, y_0, z_0) are therefore

$$\frac{x-x_0}{\dfrac{\partial F}{\partial x}\bigg|_{x_0,\,y_0,\,z_0}}=\frac{y-y_0}{\dfrac{\partial F}{\partial y}\bigg|_{x_0,\,y_0,\,z_0}}=\frac{z-z_0}{\dfrac{\partial F}{\partial z}\bigg|_{x_0,\,y_0,\,z_0}}. \qquad \text{(Arts. 220, 224.)}$$

If the equation of the surface is given in the form

$$z = f(x, y),$$

then
$$F(x, y, z) = z - f(x, y),$$

and
$$\frac{\partial F}{\partial x} = -\frac{\partial f}{\partial x} = -\frac{\partial z}{\partial x}, \quad \frac{\partial F}{\partial y} = -\frac{\partial f}{\partial y} = -\frac{\partial z}{\partial y}, \quad \frac{\partial F}{\partial z} = 1.$$

The equations of the normal then become

$$\frac{x - x_0}{\left.\dfrac{\partial z}{\partial x}\right|_{x_0, y_0, z_0}} = \frac{y - y_0}{\left.\dfrac{\partial z}{\partial y}\right|_{x_0, y_0, z_0}} = \frac{z - z_0}{-1}.$$

245. The tangent line to a space curve. Let $P(x_0, y_0, z_0)$ and $P'(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ be two points on a curve. The equations of the line through these points are

$$\frac{x - x_0}{\Delta x} = \frac{y - y_0}{\Delta y} = \frac{z - z_0}{\Delta z}. \qquad \text{(Art. 223.)}$$

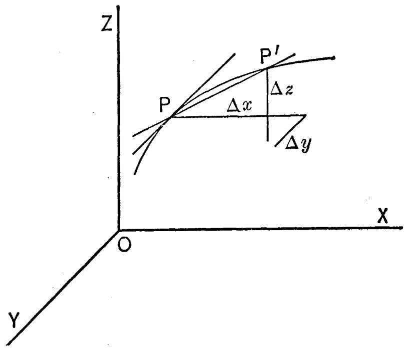


FIG. 181.

If x, y, and z are functions of some independent variable, t (compare the equations of the helix, Art. 236), Δx, Δy, and Δz will depend upon Δt. Let the above equations be multiplied by Δt. Then

$$\frac{x - x_0}{\dfrac{\Delta x}{\Delta t}} = \frac{y - y_0}{\dfrac{\Delta y}{\Delta t}} = \frac{z - z_0}{\dfrac{\Delta z}{\Delta t}}.$$

As P' approaches coincidence with P, the ratios $\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}$ approach the values of the derivatives of x, y, and z respectively at (x_0, y_0, z_0). The line through P and P' approaches at the same time the limiting position as tangent to the curve at P. Hence the equations of the tangent to the curve at (x_0, y_0, z_0) are

$$\frac{x - x_0}{\left.\frac{dx}{dt}\right|_{t_0}} = \frac{y - y_0}{\left.\frac{dy}{dt}\right|_{t_0}} = \frac{z - z_0}{\left.\frac{dz}{dt}\right|_{t_0}}. \tag{1}$$

If the equations of the curve are the simultaneous equations of two surfaces,

$$f(x, y, z) = 0,$$
$$\phi(x, y, z) = 0,$$

the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ may be obtained as follows: Since P' and P are on the surface $f(x, y, z) = 0$, therefore

$$f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) = 0,$$

and
$$f(x_0, y_0, z_0) = 0.$$

$$\therefore \frac{f(x_0 + \Delta x,\ y_0 + \Delta y,\ z_0 + \Delta z) - f(x_0,\ y_0,\ z_0)}{\Delta t} = 0.$$

Treating this expression as was done in Art. 242, there results

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0.$$

Similarly,
$$\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} = 0,$$

the values of all the derivatives being taken at the point (x_0, y_0, z_0).

From these equations there result

$$\frac{\frac{dx}{dt}}{\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial \phi}{\partial y}} = \frac{\frac{dy}{dt}}{\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial \phi}{dz}} = \frac{\frac{dz}{dt}}{\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}}.$$

Multiplying the members of eq. (1) by the corresponding members of this equation, there result as the equations of the tangent line at (x_0, y_0, z_0) to the curve whose equations are

$$f(x, y, z) = 0,$$

and

$$\phi(x, y, z) = 0,$$

$$\frac{x - x_0}{\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial \phi}{\partial y}\right)_{x_0, y_0, z_0}} = \frac{y - y_0}{\left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial z}\right)_{x_0, y_0, z_0}}$$

$$= \frac{z - z_0}{\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}\right)_{x_0, y_0, z_0}}.$$

246. Illustrations. Example 1. To find the equations of the tangent to the helix at any point.

The equations of the helix are

$$x = a \cos \theta,$$
$$y = a \sin \theta,$$
$$z = b\theta. \qquad \text{(Art. 236.)}$$

$$\therefore \frac{dx}{d\theta} = -a \sin \theta,$$

$$\frac{dy}{d\theta} = a \cos \theta,$$

$$\frac{dz}{d\theta} = b.$$

Hence the equations of the tangent to the helix at a point where $\theta = \theta_0$ are

$$\frac{x - a \cos \theta_0}{-a \sin \theta_0} = \frac{y - a \sin \theta_0}{a \cos \theta_0} = \frac{z - b\theta_0}{b}.$$

Example 2. To find the equations of the tangent to the curve of intersection of the cylinders

$$y^2 + z^2 = a^2,$$
$$x^2 + z^2 = b^2.$$

Let
$$f(x, y, z) = y^2 + z^2 - a^a,$$
$$\phi(x, y, z) = x^2 + z^2 - b^2.$$

Then
$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 2\,y, \quad \frac{\partial f}{\partial z} = 2\,z,$$
$$\frac{\partial \phi}{\partial x} = 2\,x, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 2\,z.$$

Therefore the equations of the tangent at (x_0, y_0, z_0) are
$$\frac{x - x_0}{y_0 z_0} = \frac{y - y_0}{x_0 z_0} = \frac{z - z_0}{-x_0 y_0}.$$

EXERCISE LV

Find the equation of the tangent plane to each of the following ten surfaces:

1. $x^2 + y^2 + z^2 = r^2$.
2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
4. $x^2 + y^2 = 2\,px$.
5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$.
6. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.
8. $pv = Rt$.
9. $xyz = c$.
10. $z = 3\,x + 2\,y$.
11. Prove that the direction cosines of the tangent to the helix are
$$\frac{-a \sin \theta_0}{\sqrt{a^2 + b^2}}, \quad \frac{a \cos \theta_0}{\sqrt{a^2 + b^2}}, \quad \frac{b}{\sqrt{a^2 + b^2}}.$$

(Note that the angle between the tangent and the z-axis is constant.)

12. If the point generating the helix advances in the direction of the axis of the cylinder $\frac{1}{10}$ of the radius of the cylinder at each revolution, find the angle between the tangent to the helix and an element of the cylinder.

13. Find the equations of the tangent to the helix at the point where $\theta = 30°$.

Find the equations of the tangent to the curve of intersection of each of the following pairs of surfaces.

14. $y^2 + z^2 = 1$, $x^2 + 2y^2 + 4z^2 = 4$, at a point where $z = \frac{1}{2}$.

15. $z + 2y^2 = 4$, $x^2 + y^2 - z = 0$, at $(1, 1, 2)$.

16. $z^2 + 2y^2 = 4$, $x^2 + y^2 - z^2 = 0$, at $(1, 1, \sqrt{2})$.

17. Prove that the direction cosines of the normal to

$$F(x, y, z) = 0$$

at any point (x, y, z) are

$$\frac{\dfrac{\partial F}{\partial x}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}}, \quad \frac{\dfrac{\partial F}{\partial y}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}},$$

$$\frac{\dfrac{\partial F}{\partial z}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}}.$$

18. Prove that the direction cosines of the normal to the surface

$$z = f(x, y)$$

at any point (x, y, z) are

$$\frac{\dfrac{\partial z}{\partial x}}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}}, \quad \frac{\dfrac{\partial z}{\partial y}}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}}, \quad \frac{-1}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}}.$$

TABLES

TRIGONOMETRIC FORMULAS

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A - \tan^2 A = 1$$

$$\csc^2 A - \cot^2 A = 1$$

$$\sin A \csc A = 1$$

$$\cos A \sec A = 1$$

$$\tan A \cot A = 1$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$\cot A = \frac{\cos A}{\sin A}$$

$$\sin(-A) = -\sin A$$

$$\cos(-A) = \cos A$$

$$\tan(-A) = -\tan A$$

$$\cot(-A) = -\cot A$$

$$\sec(-A) = \sec A$$

$$\csc(-A) = -\csc A$$

$$\sin\left(\frac{\pi}{2} - A\right) = \cos A$$

$$\cos\left(\frac{\pi}{2} - A\right) = \sin A$$

$$\tan\left(\frac{\pi}{2} - A\right) = \cot A$$

$$\sin\left(\frac{\pi}{2} + A\right) = \cos A$$

$$\cos\left(\frac{\pi}{2} + A\right) = -\sin A$$

$$\tan\left(\frac{\pi}{2} + A\right) = -\cot A$$

$$\sin(\pi - A) = \sin A$$

$$\cos(\pi - A) = -\cos A$$

$$\tan(\pi - A) = -\tan A$$

$$\sin(\pi + A) = -\sin A$$

$$\cos(\pi + A) = -\cos A$$

$$\tan(\pi + A) = \tan A$$

$$\sin(A + 2n\pi) = \sin A$$

$$\cos(A + 2n\pi) = \cos A$$

$$\tan(A + 2n\pi) = \tan A$$

(n a pos. or neg. integer)

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A$$
$$= 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$2\sin^2 \frac{A}{2} = 1 - \cos A$$

$$2\cos^2 \frac{A}{2} = 1 + \cos A$$

$$\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$$
$$= \frac{\sin A}{1 + \cos A}$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$
$$\cos 3A = 4\cos^3 A - 3\cos A$$
$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

A	0°	30°	45°	60°	90°	180°	270°	360°
$\sin A$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos A$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan A$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

NATURAL SINES, COSINES, AND TANGENTS

Deg.	Rad.	Sin.	Cos.	Tan.	Deg.	Rad.	Sin.	Cos.	Tan.
0	0	0	1.0000	0	**45**	.7854	.7071	.7071	1.0000
1	.0175	.0175	.9998	.0175	46	.8029	.7193	.6947	1.0355
2	.0349	.0349	.9994	.0349	47	.8203	.7314	.6820	1.0724
3	.0524	.0523	.9986	.0524	48	.8378	.7431	.6691	1.1106
4	.0698	.0698	.9976	.0699	49	.8552	.7547	.6561	1.1504
5	.0873	.0872	.9962	.0875	**50**	.8727	.7660	.6428	1.1918
6	.1047	.1045	.9945	.1051	51	.8901	.7771	.6293	1.2349
7	.1222	.1219	.9925	.1228	52	.9076	.7880	.6157	1.2799
8	.1396	.1392	.9903	.1405	53	.9250	.7986	.6018	1.3270
9	.1571	.1564	.9877	.1584	54	.9425	.8090	.5878	1.3764
10	.1745	.1736	.9848	.1763	**55**	.9599	.8192	.5736	1.4281
11	.1920	.1908	.9816	.1944	56	.9774	.8290	.5592	1.4826
12	.2094	.2079	.9781	.2126	57	.9948	.8387	.5446	1.5399
13	.2269	.2250	.9744	.2309	58	1.0123	.8480	.5299	1.6003
14	.2443	.2419	.9703	.2493	59	1.0297	.8572	.5150	1.6643
15	.2618	.2588	.9659	.2679	**60**	1.0472	.8660	.5000	1.7321
16	.2793	.2756	.9613	.2867	61	1.0647	.8746	.4848	1.8040
17	.2967	.2924	.9563	.3057	62	1.0821	.8829	.4695	1.8807
18	.3142	.3090	.9511	.3249	63	1.0996	.8910	.4540	1.9626
19	.3316	.3256	.9455	.3443	64	1.1170	.8988	.4384	2.0503
20	.3491	.3420	.9397	.3640	**65**	1.1345	.9063	.4226	2.1445
21	.3665	.3584	.9336	.3839	66	1.1519	.9135	.4067	2.2460
22	.3840	.3746	.9272	.4040	67	1.1694	.9205	.3907	2.3559
23	.4014	.3907	.9205	.4245	68	1.1868	.9272	.3746	2.4751
24	.4189	.4067	.9135	.4452	69	1.2043	.9336	.3584	2.6051
25	.4363	.4226	.9063	.4663	**70**	1.2217	.9397	.3420	2.7475
26	.4538	.4384	.8988	.4877	71	1.2392	.9455	.3256	2.9042
27	.4712	.4540	.8910	.5095	72	1.2566	.9511	.3090	3.0777
28	.4887	.4695	.8829	.5317	73	1.2741	.9563	.2924	3.2709
29	.5061	.4848	.8746	.5543	74	1.2915	.9613	.2756	3.4874
30	.5236	.5000	.8660	.5774	**75**	1.3090	.9659	.2588	3.7321
31	.5411	.5150	.8572	.6009	76	1.3265	.9703	.2419	4.0108
32	.5585	.5299	.8480	.6249	77	1.3439	.9744	.2250	4.3315
33	.5760	.5446	.8387	.6494	78	1.3614	.9781	.2079	4.7046
34	.5934	.5592	.8290	.6745	79	1.3788	.9816	.1908	5.1446
35	.6109	.5736	.8192	.7002	**80**	1.3963	.9848	.1736	5.6713
36	.6283	.5878	.8090	.7265	81	1.4137	.9877	.1564	6.3138
37	.6458	.6018	.7986	.7536	82	1.4312	.9903	.1392	7.1154
38	.6632	.6157	.7880	.7813	83	1.4486	.9925	.1219	8.1443
39	.6807	.6293	.7771	.8098	84	1.4661	.9945	.1045	9.5144
40	.6981	.6428	.7660	.8391	**85**	1.4835	.9962	.0872	11.4301
41	.7156	.6561	.7547	.8693	86	1.5010	.9976	.0698	14.3007
42	.7330	.6691	.7431	.9004	87	1.5184	.9986	.0523	19.0811
43	.7505	.6820	.7314	.9325	88	1.5359	.9994	.0349	28.6363
44	.7679	.6947	.7193	.9657	89	1.5533	.9998	.0175	57.2900
					90	1.5708	1.0000	0	∞

ANSWERS TO PROBLEMS

Exercise V

2. $36° \, 53'$, $143° \, 7'$.
3. $24° \, 31'$, $204° \, 31'$.
4. $27° \, 46'$, $152° \, 14'$, etc.; 9560; .35.

Exercise VI

3. $(7.62, -66° \, 44')$, $(5, 36° \, 52')$.
4. $(1.73, 1)$, $(-2.12, -2.12)$.
6. $\theta = 90°$; $\theta = 0$; $r \cos \theta = c$; $\theta = 45°$; $\theta = 135°$.
8. $y = 0$; $x = y\sqrt{3}$; $y = cx$; $x^2 + y^2 = 25$; $x^2 + y^2 = c^2$.
9. $(4.21, -39° \, 19')$.
10. $(5.24, 3.57)$.

Exercise VII

3. 7.62, $\sqrt{x^2 + y^2}$. **4.** 11.40. **6.** $x^2 + y^2 = 25$.

Exercise VIII

1. 5.04. **2.** 11.65. **3.** $\sqrt{(a-c)^2 + (b-d)^2}$. **4.** 7.47. **5.** 5.

Exercise IX

1. 8.06. **2.** 9.90.
3. $AB = 7.07$, $BC = 8.36$, $CA = 7.70$, $OA = 3.91$, $OB = 4.27$, $OC = 5.15$.
4. 5.95.
5. 12.73, 14.87, 2.24.
6. 5.99, 5.54, 6.40.

Exercise X

3. $-1:5$, $3:1$, $-7:3$.

Exercise XI

1. $(\frac{16}{5}, -\frac{9}{5})$.
2. $(-29, 27.5)$, $(27, -18)$.
3. $r = 2$, $k = -\frac{11}{3}$.
4. $\left(\dfrac{a - nc}{1 - n}, \dfrac{b - nd}{1 - n}\right)$.
5. $\left(\dfrac{x_1 + 2\,x_2}{3}, \dfrac{y_1 + 2\,y_2}{3}\right)$, $\left(\dfrac{2\,x_1 + x_2}{3}, \dfrac{2\,y_1 + y_2}{3}\right)$.

Exercise XII

2. $26° 34'$, $63° 26'$.
3. $(-\frac{1}{7}, 0)$.
6. $(12, -1)$, or $(-6, -19)$, or $(2, 9)$.
7. $139° 24'$. **8.** $42° 50'$.
10. -6, $99° 25'$. **12.** $\frac{2m}{1-m^2}$.
13. $87° 4'$. **14.** $-.3332$.
15. $(1.2, -4.56)$. **16.** $(0, -am)$.
17. $\frac{m+n}{1-mn}$, $\left(0, \frac{-a(m+n)}{1-mn}\right)$.

Exercise XIII

1. 71.5. **2.** 22.56.
3. $\frac{1}{2}(x_1y_2 - x_2y_1)$.
4. $\frac{1}{2} r_1r_2 (\sin \theta_2 - \theta_1)$.
5. 160.

Exercise XIV

1. 185. **2.** 3684.34 sq. ft.
3. 60305 sq. ft.
4. (*d*) $41° 3'$, $38° 27'$, $100° 30'$.
12. 13.5. **13.** $x + y = 6$.
14. $x^2 + y^2 - 4x - 6y = 12$.
16. $114° 19'$. **17.** -4.871.
18. -4.186.
19. $\frac{a+b}{1-ab}$. **20.** 3.154.

Exercise XV

1. $3x + 7y = 31$.
4. $2x - y - 11 = 0$.
5. $\frac{x}{a} + \frac{y}{b} = 1$.
6. $y = mx + b$.
7. $x^2 + y^2 - 4x + 8y = 5$; Intercepts, $x = 5$ or -1; $y = .58$ or -8.58.
10. $(x-h)^2 + (y-k)^2 = r^2$.
11. $x^2 + y^2 = r^2$.
12. $x^2 + y^2 \pm 2rx \pm 2ry + r^2 = 0$.
14. $16x^2 + 7y^2 = 112$.
15. $4x^2 - 5y^2 + 20 = 0$.
16. $x^2 + 8y + 16 = 0$.
17. $63x^2 + 143y^2 - 18xy + 216x - 456y - 1728 = 0$.
18. $52x^2 - 80y^2 + 224xy - 68x + 496y - 1343 = 0$.
19. $y^2 + 22x - 8y - 39 = 0$.

Exercise XVIII

1. $x^2 + 4y^2 = 18$.
2. $64x^2 - 64y^2 + 3 = 0$.
3. $4x^2 + y^2 = 12$.
6. Lines $x + 2y = 0$ and $2x - 3y = 0$, referred to ∥ axes through $(1, 1)$.
7. $x^2 = -\frac{7\sqrt{2}}{2}y$, $\theta = 45°$, new origin $(1.77, .93)$.
8. Lines $x - 2y = 0$, and $x + y = 0$, referred to ∥ axes through $(0, -1)$.
9. $2x^2 + y^2 = 4$, $\theta = 45°$.
10. $x^2 - y^2 = 8$, $\theta = 45°$.

Exercise XIX

1. $2x + y = 5.$
3. $8x - 3y = 24.$
7. $3x + 2y \pm 5\sqrt{13} = 0.$
8. $3x - y = 11.$
9. $bx - ay = 0.$
10. $y - y_1 = m(x - x_1).$
11. $Bx - Ay + Ab = 0.$
20. $y - 2 = 6.94(x - 1).$
21. $L_1,\ 4x + 3y + 18 = 0;$
 $L_2,\ 17x - 6y + 39 = 0.$
22. $3x + 4y + 75 = 0.$
26. $y - k = \dfrac{l + m}{1 - lm}(x - h).$
27. $-.5642x + .8257y = 3,$
 $.9780x + .2088y = 3.$
31. $3x + y + 10 = 0.$
33. $107x + 134y - 187 = 0.$
34. $5x + 5y = 12.$
35. $118x + 177y = 486.$
36. $63x + 147y = 536.$

Exercise XX

1. 3.84.
2. .383.
5. 1.06.
6. 3.13.
7. $\dfrac{mx_1 - y_1 + b}{\pm\sqrt{m^2 + 1}}.$
8. $x_0 \cos\alpha + y_0 \sin\alpha - p.$

Exercise XXI

1. $1.23\, r \sin\theta - .134\, r \cos\theta = 1.$
5. $(8.94,\ 26° 34').$
6. $(4.91,\ 102° 50').$

Exercise XXII

1. $(0, 0),\ r = 5.$
2. $(2, -3),\ r = 5.$
4. $(-.75,\ 1.75),\ r = 3.02.$
5. $(1, -2),\ r = 0.$
7. $(-.5, -.5),\ r = .707.$
8. No locus.
9. $(2, -3),\ r = 5.10.$
12. $\left(\dfrac{a}{2},\ \dfrac{b}{2}\right),\ r = \dfrac{1}{2}\sqrt{a^2 + b^2}.$
13. $x^2 + y^2 + 2x - 6y + 6 = 0.$
15. $x^2 + y^2 + 2x - 6y = 35.$
16. $x^2 + y^2 - 4x - 10y + 20 = 0.$
17. $x^2 + y^2 + 6x + 12y = 85.$
18. $x^2 + y^2 - 11x - 17y + 30 = 0.$
19. $27(x^2 + y^2) - 65x + 15y$
 $- 250 = 0.$
20. $(x - \frac{17}{4})^2 + (y + \frac{3}{2})^2 = (\frac{15}{14})^2.$
21. $(x - \frac{29}{8})^2 + (y + \frac{27}{8})^2 = (\frac{15}{8})^2.$

Exercise XXIII

1. $x^2 = \frac{1}{2}y,\ V(\frac{3}{4},\ \frac{31}{8}).$
3. $x^2 = y,\ V(-2,\ 0).$
4. $x^2 = \frac{5}{2}y,\ V(1,\ \frac{1}{5}).$
6. $y^2 = -\frac{4}{3}x,\ V(\frac{5}{6},\ \frac{1}{3}).$
9. 19.44 ft., 17.78 ft., 15 ft.,
 11.11 ft., 6.11 ft.
10. $5y = x^2 + x - 2.$
12. $2h^2y = (y_1 + y_3 - 2y_2)x^2 + h(y_3 - y_1)x + 2h^2y_2.$

Exercise XXIV

1. $x^2 = 4y$, $V(3, -2)$.
2. $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $C(2, 3)$.
3. The lines $3x - y + 7 = 0$, $3x + y + 5 = 0$.
4. $\frac{x^2}{1} - \frac{y^2}{16} = 1$, $C\left(\frac{5}{2}, -3\right)$.
8. $\frac{x^2}{1} + \frac{y^2}{\frac{9}{5}} = 1$, $\theta = \tan^{-1} 2$.
9. $\frac{x^2}{9.47} - \frac{y^2}{3.99} = 1$, $\theta = 19° 47'$.
10. $y^2 = 4x$, $\theta = \tan^{-1} \frac{4}{3}$; $V(-3, -3)$.
11. $x^2 - y^2 = 16$, $\theta = 45°$.
12. $(\sqrt{2} - 1)x^2 - (\sqrt{2} + 1)y^2 = 10$, $\theta = 67° 30'$.
13. $8x^2 + 28y^2 = 13$, $C(-\frac{1}{2}, \frac{1}{2})$, $\theta = \tan^{-1} \frac{1}{2}$.
14. The lines $2x + 7y = 0$, $7x - 2y = 0$, referred to $\parallel$ axes through $(-\frac{3}{53}, \frac{16}{53})$.
15. $y^2 - x^2 = 2$, $C(-1, -2)$, $\theta = 45°$.
18. $y - k = \pm \frac{b}{a}(x - h)$.
19. $x = b$ and $y = a$.
23. $xy - 4x - 2y + 12 = 0$.

Exercise XXVIII

8. $\pm 3\sqrt{5}$.
9. $\pm r\sqrt{1 + m^2}$.
10. 6.

Exercise XXX

1. $y = \sqrt{3}x \pm 8$.
2. $y = x \pm 10$.
3. $x + 2y + 6 = 0$.
6. $x - 2y + 6 = 0$, $3x - 2y + 2 = 0$.
12. $2x + 2y + p = 0$.
16. $y = \frac{\sqrt{2}}{4}x + 3\sqrt{2}$, $y = -\frac{\sqrt{2}}{4}x - 3\sqrt{2}$.
17. $y = 2x - 7 \pm 2\sqrt{10}$.

Exercise XXXI

1. .007651, .030301, 3.003001.
2. -1, $-\frac{1}{2}$, $\frac{3}{2}$.
3. $\frac{3}{8}$, $3\frac{3}{8}$, 0, $\frac{3}{2}$.
4. $4x - 4y = 5$.

Exercise XXXII

1. $x + y + 1 = 0$, $x - y = 3$.
2. $3x - 4y + 25 = 0$, $4x + 3y = 0$.
3. Tangents, $2y_0y - 3x_0^2x + x_0^3 = 0$, $y = 0$, $3x - 2y = 1$, $3x - y = 4$.
5. Tangents, $6x - y = 6$, $6x + y + 30 = 0$.
8. $12° 6'$, $36° 52'$.
9. $73° 41'$.
10. $(-5, -6.75)$.

Exercise XXXIII

1. $2x - \dfrac{1}{2\sqrt{x}} - 3.$

2. $2ax - \dfrac{2c}{x^3}.$

3. $\dfrac{x}{\sqrt{x^2 + a^2}}.$

4. $\dfrac{-2a}{(t-a)^2}.$

5. $\dfrac{2x^2+1}{\sqrt{x^2+1}}.$

6. $\dfrac{i}{\sqrt{i^2 - a^2}} + 1.$

7. $2t + \dfrac{2}{t^3}.$

8. $\dfrac{2x}{x^2+1} \cdot \dfrac{1}{\sqrt{x^4 - 1}}.$

9. $(x+a)^{n-1}(x+b)^{m-1}[x(n+m) + ma + nb].$

10. $\dfrac{-an}{s^{n+1}}.$

11. $2anx(ax^2+b)^{n-1}.$

12. $-2an \cdot \dfrac{(x+a)^{n-1}}{(x-a)^{n+1}}.$

13. $y = mx + b.$

14. $8x - y = 4.$

15. $x - y + 1 = 0.$

16. $4x - 3y + 25 = 0.$

21. $\dfrac{1}{2}(y+y_0) = ax_0x + \dfrac{b}{2}(x+x_0) + c.$

22. $\dfrac{1}{2}(x+x_0) = ay_0y + \dfrac{b}{2}(y+y_0) + c.$

Exercise XXXIV

1. $(b-a)\sin 2x.$

2. $24\tan^2 2x(1+\tan^2 2x).$

3. $\frac{3}{2}\cos t\sqrt{\sin t}.$

4. $-2\sin 2x.$

5. $-a\sin 2(ax+b).$

6. $\dfrac{2\sin x}{\cos^3 x}.$

7. $\dfrac{12\sin 3x}{\cos^5 3x}.$

8. $\dfrac{2(1+\sin t)\sec^2 2t - \cos t\tan 2t}{(1+\sin t)^2}.$

9. $\dfrac{\sin x}{2\cos\frac{3}{2}x}(1+3\cos^2 x).$

10. $nm(\tan^{n-1} mx + \tan^{n+1} mx).$

11. $\dfrac{2(\sin^4 x + \cos^4 x)}{\sin^3 x\cos^3 x}.$

12. $-\dfrac{\cos x}{2\sin\frac{3}{2}x}.$

13. $\tan^4\theta.$

14. $x\cos x + \sin x.$

15. $x\sec^2 x + \tan x.$

16. $\sin x + x\cos x.$

17. $4\csc 4x(1 - 2\csc^2 4x).$

18. $-mnq\dfrac{\cos^{n-1} qx}{\sin^{n+1} qx}.$

19. $(x+1)\sin x + (x-1)\cos x$

20. $abn\sin^{n-1} bt\cos bt.$

Exercise XXXVII

1. $y^2 - 6y + 8x = 23.$

2. $\dfrac{x^2}{6} + \dfrac{(y+1)^2}{15} = 1.$

3. $3x^2 - y^2 - 16x + 8y = 0.$

Exercise XLI

1. No locus.
2. Hyperbola.
3. Two intersecting lines.
4. Two parallel lines.
5. One line.
6. Ellipse.
7. A point.
8. No locus.
9. Parabola.
10. Hyperbola.

Exercise LIII

2. $x = \frac{a}{h}(h - b\theta)\cos\theta, \quad y = \frac{a}{h}(h - b\theta)\sin\theta, \quad z = b\theta.$
3. $x = \sqrt{a^2 - b^2\theta^2}\cdot\cos\theta, \quad y = \sqrt{a^2 - b^2\theta^2}\cdot\sin\theta, \quad z = b\theta.$
4. $r = \frac{a}{h}(h - b\theta).$
5. $r = \sqrt{a^2 - b^2\theta^2}.$

Exercise LV

1. $x_0x + y_0y + z_0z = r^2.$
2. $\frac{x_0x}{a^2} - \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = 1.$
4. $x_0x + y_0y = p(x + x_0).$
5. $\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = \frac{z + z_0}{2c}.$
6. $\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1.$
8. $p_0v + v_0p = R(t + t_0).$
9. $y_0z_0x + x_0z_0y + x_0y_0z = 3c.$
14. $\frac{2x - \sqrt{6}}{2\sqrt{3}} = \frac{2y - \sqrt{3}}{\sqrt{6}} = \frac{2z - 1}{-3\sqrt{2}}$; three other answers.
15. $\frac{x - 1}{3} = \frac{y - 1}{-1} = \frac{z - 2}{4}.$

INDEX

The numbers refer to the pages.